Math 230 **Spring 2019**
Exam 1 — Thurs Feb 21 **Drew Armstrong**

Problem 1. Logic.

(a) Complete the following truth table.

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

(b) Use a truth table to prove that $\neg(P \Rightarrow Q) = P \wedge \neg Q$.

P	Q	$\neg Q$	$P \wedge \neg Q$	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	F	T	F

(c) Use (b) to find the **opposite** of the following statement:

"For all $x \in S$ we have $P(x) \Rightarrow Q(x)$."

From de Morgan's law, the opposite statement is

"There exists some $x \in S$ such that $\neg(P(x) \Rightarrow Q(x))$."

From part (b) this is the same as

"There exists some $x \in S$ such that $P(x)$ and $\neg Q(x)$."

Problem 2. Applying Logic.

(a) Use the contrapositive to prove for all $n \in \mathbb{Z}$ that "if n^2 is even then n is even."

Proof. For any $n \in \mathbb{Z}$ we will prove that "if n is odd then n^2 is odd." So assume that n is odd, which means that $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then we have

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2) + 1,$$

which is odd. $\square$

(b) Use the method of contradiction to prove that $\sqrt{18} \notin \mathbb{Q}$. You may assume that $\sqrt{2} \notin \mathbb{Q}$.

Proof. Assume for contradiction that $\sqrt{18} = a/b$ for some integers $a, b \in \mathbb{Z}$. But then since $\sqrt{18} = 3\sqrt{2}$ we have

$$\sqrt{2} = \frac{1}{3}\sqrt{18} = \frac{1}{3} \cdot \frac{a}{b} = \frac{a}{3b} \in \mathbb{Q},$$

which contradicts the fact that $\sqrt{2} \notin \mathbb{Q}$. $\square$

(c) Use the principle from 1(c) to prove that the following statement is **false**:

"For all integers $a, b, c \in \mathbb{Z}$ we have $(ab = ac) \Rightarrow (b = c)$."

Proof. According to 1(c), the opposite of this statement is

"There exist some integers $a, b, c \in \mathbb{Z}$ such that $ab = ac$ and $b \neq c$."

To prove this it suffices to give a specific example. Take $a = 0$, $b = 1$ and $c = 2$. Then we have $ab = ac$ and $b \neq c$. $\square$

Problem 3. Divisibility.

(a) For integers $a, b \in \mathbb{Z}$, state the formal definition of "$a|b$."

$$\text{"}a|b\text{"} = \text{"}\exists k \in \mathbb{Z}, ak = b.\text{"}$$

(b) For all integers $n \in \mathbb{Z}$, prove that $n|0$ and $1|n$.

Proof. To see that $n|0$ note that $n \cdot 0 = 0$. To see that $1|n$ note that $1 \cdot n = n$. $\square$

(c) For all integers $a, b, c \in \mathbb{Z}$, prove that "if $a|b$ and $a|c$ then $a|(b + c)$."

Proof. Assume that $a|b$ and $a|c$. By definition this means that we have $ak = b$ and $a\ell = c$ for some integers $k, \ell \in \mathbb{Z}$. But then we have

$$b + c = ak + a\ell = a(k + \ell),$$

which implies that $a|(b + c)$. $\square$

Problem 4. Induction. For any integer $n \in \mathbb{Z}$ we define the following statement:

$$P(n) := \text{"There exists an integer } k \in \mathbb{Z} \text{ such that } 4^n = 3k + 1.\text{"}$$

(a) Prove that the statements $P(0)$, $P(1)$ and $P(2)$ are true.

Proof. To see that $P(0), P(1), P(2)$ are true, take $k = 0, 1, 5$, respectively. $\square$

(b) Now fix some integer $n \geq 0$ and assume for induction that $P(n)$ is true. In this case, prove that $P(n+1)$ is also true.

Proof. Since $P(n)$ is true we can write $4^n = 3k+1$ for some $k \in \mathbb{Z}$. But then we have

$$\begin{aligned} 4^{n+1} &= 4 \cdot 4^n \\ &= 4 \cdot (3k+1) \\ &= 4 \cdot 3k + 4 \\ &= 4 \cdot 3k + 3 + 1 \\ &= 3(4k+1) + 1, \end{aligned}$$

which implies that $P(n+1)$ is also true. $\square$

(c) What do you conclude from this?

From the Principle of Induction I conclude that $P(n)$ is true for **all** integers $n \geq 0$.