Math 230 **Fall 2018**
Homework 2 **Drew Armstrong**

Problem 1. De Morgan's Laws say that for all statements P, Q we have

$$\neg(P \vee Q) = \neg P \wedge \neg Q \qquad \text{and} \qquad \neg(P \wedge Q) = \neg P \vee \neg Q.$$

(a) Use truth tables to prove these laws.
(b) Use a truth table to prove that $(P \Rightarrow Q) = (\neg P) \vee Q$ for all statements P, Q.
(c) Combine parts (a) and (b) to prove that for all statements P, Q we have

$$(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P).$$

Do **not** use a truth table.

Problem 2. Practice with logical analysis.

(a) Use the results of Problem 1 to prove that for all statements P, Q, R we have

$$P \Rightarrow (Q \vee R) = (\neg Q \wedge \neg R) \Rightarrow \neg P.$$

Do **not** use a truth table.
(b) Use the result of (a) to prove that for all $a, b, c, d \in \mathbb{Z}$ we have

$$a + b \leq c + d \quad \Longrightarrow \quad a \leq c \quad \text{or} \quad b \leq d.$$

(c) Prove that the converse of the statement in part (b) is **false**. [Hint: To prove that a universal statement is false it is enough to provide a single counterexample.]

Problem 3. I will guide you through an induction proof that

$$(a-1)\Big|(a^n - 1) \qquad \text{for all integers } a, n \in \mathbb{Z} \text{ such that } n \geq 1.$$

For the purpose of the proof, let $a \in \mathbb{Z}$ be a fixed integer. We will use induction on n.

(a) Prove that $(a-1)\big|(a^n - 1)$ when $n = 1$.
(b) Now assume that $(a-1)\big|(a^n - 1)$ is true for some fixed $n \geq 1$. In this case, prove that

$$(a-1)\Big|(a^{n+1} - 1).$$

Problem 4. For all integers $d \in \mathbb{Z}$ let us define the statement

$$P(d) := \text{“ } \forall n \in \mathbb{Z}, d|n^2 \Rightarrow d|n. \text{ ”}$$

(a) Now fix an integer $d \geq 1$ and prove that

$$P(d) \Longrightarrow \sqrt{d} \notin \mathbb{Q}$$

[Hint: Mimic the proofs from class when $d = 2$ and $d = 3$.]
(b) Prove that $P(5)$ is a true statement, and hence that $\sqrt{5}$ is irrational.
(c) Prove that $P(12)$ is a false statement. [Remark: It is still true that $\sqrt{12}$ is irrational, but the method of proof from part (a) will not work. Maybe you can see how to fix it.]