Math 230 **Fall 2018**
Exam 1 — Thurs Sept 20 **Drew Armstrong**

Problem 1.

(a) State the *principle of the contrapositive.*

For all statements P, Q we have

$$(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P).$$

(b) State *de Morgan's law.*

For all statements P, Q we have

$$\neg(P \vee Q) = \neg P \wedge \neg Q \quad \text{and} \quad \neg(P \wedge Q) = \neg P \vee \neg Q.$$

(c) Explicitly use these two principles to prove for all statements P, Q, R that

$$P \Rightarrow (Q \vee R) = (\neg Q \wedge \neg R) \Rightarrow \neg P.$$

Do **not** use a truth table.

Proof. For all statements P, Q, R we have

$$\begin{aligned} P \Rightarrow (Q \vee R) &= \neg(Q \vee R) \Rightarrow \neg P, && \text{from (a)} \\ &= (\neg Q \wedge \neg R) \Rightarrow \neg P. && \text{from (b)} \end{aligned}$$

Problem 2. Let $m, n \in \mathbb{Z}$ be any integers.

(a) If either m or n is even, prove that mn is even.

Proof 1. There are two cases. (Case 1) If m is even then we can write $m = 2k$ for some $k \in \mathbb{Z}$. It follows that $mn = (2k)n = 2(kn)$ is even. (Case 2) If n is even then we can write $n = 2\ell$ for some $\ell \in \mathbb{Z}$ It follows that $mn = m(2\ell) = 2(m\ell)$ is even. □

[Remark: Here I have used the principle $(P \vee Q) \Rightarrow R = (P \Rightarrow R) \wedge (Q \Rightarrow R)$.]

Proof 2. Without loss of generality, let us assume that m is even. Then we can write $m = 2k$ for some $k \in \mathbb{Z}$ and it follows that $mn = (2k)n = 2(kn)$ is even. □

(b) If mn is even prove that either m or n is even. [Hint: 1(c).]

Proof. Consider the statements $P =$"mn is even," $Q =$"m is even" and $R =$"n is even." We are asked to prove that $P \Rightarrow (Q \vee R)$, which by 1(c) is equivalent to the statement $(\neg Q \wedge \neg R) \Rightarrow \neg P$. In other words, we are asked to prove:

"if m and n are both odd then mn is odd."

So let us assume that m and n are both odd. This means that there exist $k, \ell \in \mathbb{Z}$ such that $m = 2k + 1$ and $n = 2\ell + 1$. It follows that

$$mn = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1,$$

which is odd. □

Problem 3.

(a) For integers $a, b \in \mathbb{Z}$, state the formal definition of "$a|b$."

$$\text{“}\, a|b \,\text{”} = \text{“}\, \exists k \in \mathbb{Z}, ak = b \,\text{”}$$

(b) For all $n \geq 1$ consider the statement $P(n) =$ "$3|(4^n - 1)$." Prove that $P(1)$ is true.

The statement is $P(1) =$ "$3|3$." This statement is true because $3 \cdot 1 = 3$ and $1 \in \mathbb{Z}$.

(c) Now consider any positive integer $n \geq 1$ and assume for induction that $P(n)$ is true. In this case prove that $P(n+1)$ is also true.

Proof. If $P(n)$ is true then there exists $k \in \mathbb{Z}$ such that $3k = 4^n - 1$. But then we have

$$\begin{aligned} 4(3k) &= 4(4^n - 1) \\ 12k &= 4^{n+1} - 4 \\ 12k + 3 &= 4^{n+1} - 1 \\ 3(4k+1) &= 4^{n+1} - 1, \end{aligned}$$

which implies that $P(n+1)$ is true. □

Problem 4. Consider the followiing statement:

"For all $n \in \mathbb{Z}$, if $3 \nmid n$ then there exists $k \in \mathbb{Z}$ such that $n = 3k + 1$."

(a) Prove that this statement is false.

Proof. I claim that $n = 2$ is a counterexample. Indeed, we have $3 \nmid 2$, but there does not exist $k \in \mathbb{Z}$ such that $2 = 3k + 1$. In other words, for all $k \in \mathbb{Z}$ we have $2 \neq 3k + 1$. □

(b) Write down the correct version: " $\forall n \in \mathbb{Z}$, if $3 \nmid n$ then $\exists k \in \mathbb{Z}$ such that ? "

$$n = 3k + 1 \quad \text{or} \quad n = 3k + 2.$$

(c) Use the correct version to prove that for all $n \in \mathbb{Z}$ we have $3|n^2 \Rightarrow 3|n$.

Proof. We will prove the contrapositive statement that $3 \nmid n \Rightarrow 3 \nmid n^2$ for all $n \in \mathbb{Z}$. So consider any $n \in \mathbb{Z}$ and assume that $3 \nmid n$. From part (b) this means that $n = 3k + 1$ or $n = 3k + 2$ for some $k \in \mathbb{Z}$. In the first case we have

$$n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$$

and in the second case we have

$$n^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1.$$

In either case we conclude that $3 \nmid n^2$. □