Math 230 D **Fall 2015**
Homework 3 **Drew Armstrong**

Problem 1. How do $-$ and $\times$ interact? For the following exercises I want you to give Euclidean style proofs using the axioms of $\mathbb{Z}$ from the handout. You can also use the results we proved in class, such as: uniqueness of "$-a$", $0a = 0$ for all $a \in \mathbb{Z}$, and the Cancellation Lemma $(a + b = a + c) \Rightarrow (b = c)$.

(a) Recall that $-n$ is the **unique** integer satisfying $n + (-n) = 0$. Prove that for all $n \in \mathbb{Z}$ we have $-(-n) = n$.
(b) Prove that for all $a, b \in \mathbb{Z}$ we have $(-a)b = a(-b) = -(ab)$. [Hint: Use the fact that $0a = 0$ for all $a \in \mathbb{Z}$, which we proved in class.]
(c) Recall that for all $m, n \in \mathbb{Z}$ we define $m - n := m + (-n)$. Prove that for all $a, b, c \in \mathbb{Z}$ we have $a(b - c) = ab - ac$. [Hint: Use (b).]
(d) Prove that for all $a, b \in \mathbb{Z}$ we have $(-a)(-b) = ab$. [Hint: Show that $-(ab) = a(-b)$. Then use (a) and (b).]

Problem 2. First Look at Induction.

(a) Prove that 3^n is an odd number **for all** natural numbers $n \in \mathbb{N}$. [Hint: Assume for contradiction that **there exists** a natural number such that 3^n is **even**. In this case, the Well-Ordering Axiom tells us that there is a **smallest** such integer. Call it $m \in \mathbb{N}$. Now try to find a contradiction.]
(b) Assume that there exists a real number $x \in \mathbb{R}$ such that $2^x = 3$ (we call it $x = \log_2(3)$). Use part (a) to prove that $x \notin \mathbb{Q}$.

Problem 3. Square root of $a \in \mathbb{Z}$.

(a) Suppose that $\alpha \in \mathbb{R}$ and $\alpha \notin \mathbb{Z}$. In this case, use the Well-Ordering Axiom to prove that there exists an integer $b \in \mathbb{Z}$ such that

$$b < \alpha < b + 1.$$

[Hint: Let $S = \{n \in \mathbb{Z} : \alpha < n\}$. Since this set is nonempty and bounded below, the Well-Ordering Axiom says it has a least element, say $m \in S$.]

(b) Prove that for all $a \in \mathbb{Z}$ we have

$$\sqrt{a} \notin \mathbb{Z} \Longrightarrow \sqrt{a} \notin \mathbb{Q}.$$

[Hint: Assume that $\sqrt{a} \notin \mathbb{Z}$, so we have $b < \sqrt{a} < b + 1$ for some $b \in \mathbb{Z}$ by part (a). Now assume for contradiction that $\sqrt{a} \in \mathbb{Q}$. Consider the set $T = \{n \in \mathbb{N} : n\sqrt{a} \in \mathbb{Z}\}$. Show that T is not empty, so by Well-Ordering it has a smallest element, say $m \in T$. Now show that $m(\sqrt{a} - b)$ is a **smaller** element of T. Contradiction.]

Problem 4. Greatest Common Divisor. Consider two integers $a, b \in \mathbb{Z}$ that are not both zero. Now consider the set of "common divisors"

$$D = \{d \in \mathbb{Z} : d|a \wedge d|b\}.$$

Show that this set is bounded above, so by Well-Ordering it has a largest element. Call the largest element $\gcd(a, b)$. Now show that $1 \leq \gcd(a, b)$. [Hint: Use Problem 3(d) from HW1.]