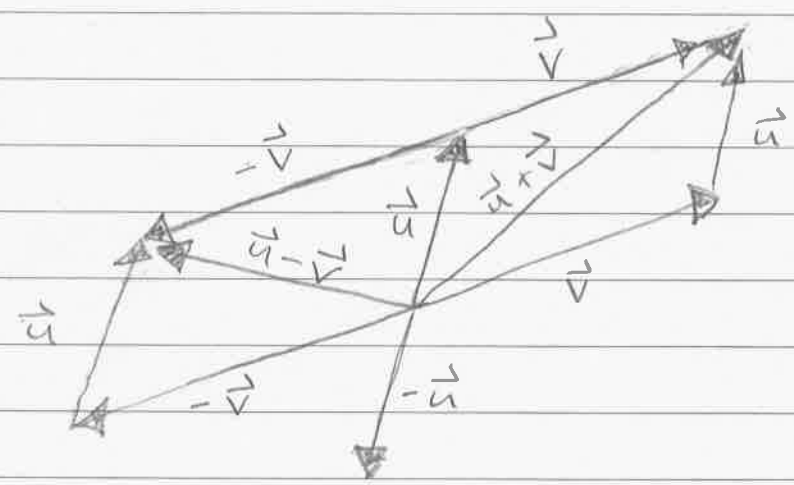
9/9/15

HW1 due now.
HW2 will be due Wed Sept 23.
Exam 1 will be Fri Sept 25.

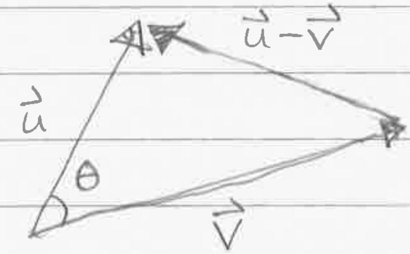
Discuss HW1. Any questions?

I would like to discuss HW1.4.

Recall: If $\vec{u}$ and $\vec{v}$ are two vectors in the plane then we can add them and subtract them like this:



Note that we can move the vector $\vec{u}-\vec{v}$ and place it like so to form a triangle:



What can we say about the angle θ?

On Problem 4 you used the Pythagorean Theorem and its converse to prove that

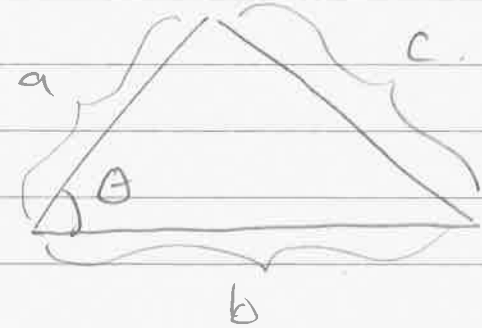
$$\boxed{\theta = 90^\circ \iff \vec{u}\cdot\vec{v} = 0}$$

But what can we say about the dot product when $\theta \neq 90^\circ$.

You may have seen the following generalization of the Pythagorean Theorem.

☆ The Law of Cosines:

Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of length a and b.



Then I claim that

$$c^2 = a^2 + b^2 - 2ab\cos\theta .$$

[Question: Why do I call this a "generalization" of the Pythagorean Theorem.]

There are many elementary proofs of this but I won't give one right now. Instead, let's apply the Law of Cosines to our vector triangle from before. We get

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta .$$

On the other hand, a purely algebraic calculation [Problem 4(b)] tells us that

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v}).$$

Equating right hand sides gives

$$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2(\vec{u}\cdot\vec{v})$$
$$= \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta.$$

$$-\cancel{2}(\vec{u}\cdot\vec{v}) = -\cancel{2}\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta}$$

This is a very beautiful formula. It relates something purely algebraic (on the left) to something purely geometric (on the right).

In fact, this is quite deep. In modern mathematics we define geometry via algebra and we use the dot product to do it.

Q: What is n-dimensional space?

A: Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}^n$ be the set of ordered n-tuples of real numbers.

[So $\mathbb{R}^2$ is the Cartesian plane.]

We will think of n-tuples of real numbers as "points in n-dimensional space".

Given two points $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\vec{v} = (v_1, v_2, \cdots, v_n)$ we define their <u>dot product</u>

$$\vec{u} \circ \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

We <u>define</u> the length $\|\vec{u}\|$ of the vector $\vec{u}$ by

$$\|\vec{u}\|^2 := \vec{u} \cdot \vec{u}$$
$$= u_1^2 + u_2^2 + \cdots + u_n^2.$$

$$\Longrightarrow \|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

This allows us to compute the distance $\|\vec{u}-\vec{v}\|$ between points $\vec{u}$ and $\vec{v}$:

$$\|\vec{u}-\vec{v}\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}.$$

We can also define the angle between vectors $\vec{u}$ and $\vec{v}$. Remember our formula from 2D geometry:

$$\vec{u}\circ\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

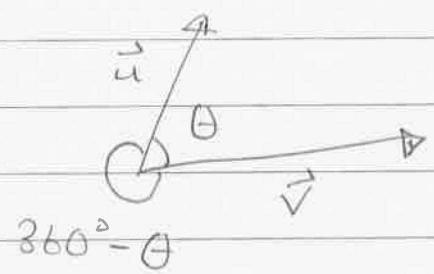
$$\frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \cos\theta$$

$$\Longrightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right).$$

We will just assume that this formula still holds in higher dimensional geometry. Given two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ we will define

$$\text{angle}(\vec{u},\vec{v}) := \cos^{-1}\left(\frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$

[Is there a possible issue here? After all, there are two <u>different</u> angles between the vectors $\vec{u}$ and $\vec{v}$:



Yes, but the angles θ and $360° - \theta$ have the same cosine so there's no problem.]

Now we can "do" n-dimensional geometry, even if we can't "see" n-dimensional geometry.

<u>Definition</u>:

The set $\mathbb{R}^n$ together with the dot product operation is called

"n-dimensional Euclidean space".

[What would Euclid think?!]

9/11/15

HW 2 will be assigned Monday
and due on Wed Sept 23
Exam 1 is on Fri Sept 25 in class.

I think we've seen enough geometry for now. Our next mathematical topic will be number theory. But there are also logical issues to discuss.

One of the most important methods of proof is called "proof by contradiction". As an example, I will prove the second oldest theorem in mathematics (after the Pythagorean Theorem).

☆ Theorem: The square root of 2 is not a ratio of whole numbers.

Proof: Assume for contradiction that $\sqrt{2}$ is a ratio of whole numbers. In this case, we can write

$$\sqrt{2} = \frac{a}{b}$$

in "lowest terms"

$\downarrow$

(i.e., where a and b are whole numbers with no common factors except ± 1).

Now we can square both sides to get

$$2 = \frac{a^2}{b^2}$$

$$\implies a^2 = 2b^2.$$

This implies that a^2 is even, and hence a is even (as we proved last week). That is, there exists a whole number k such that $a = 2k$. Substituting this into our equation gives

$$\begin{aligned} a^2 &= 2b^2 \\ (2k)^2 &= 2b^2 \\ 4k^2 &= 2b^2 \\ 2k^2 &= b^2. \end{aligned}$$

This implies that b^2 is even, and hence b is even, i.e., there exists a whole number ℓ such that $b = 2\ell$.

Since $a = 2k$ and $b = 2l$ we conclude that a and b have common factor 2.

But this is impossible because we already know that a and b have no common factors except ± 1.

Since our original assumption (that $\sqrt{2}$ is a ratio of whole numbers) leads to a contradition, we conclude that it was false, i.e., $\sqrt{2}$ is not a ratio of whole numbers.

//

What do you you make of that proof? Do you find it convincing?

Let's discuss the logic behind it. In this class our logic will follow two rules.

Rule 1 ("excluded middle")

Any given mathematical statement is either T or F (not both and not neither).

Statements without this property are not "mathematical statements".

Examples:

- "$0 = 1$" is a math. statement
- "Today is a nice day" is <u>not</u>.

Rule 2 ("material implication")

T flows along arrows $\Longrightarrow$.

In other words,

$T \Rightarrow T$, $F \Rightarrow T$, $F \Rightarrow F$, $T \Rightarrow F$.

✓ ✓ ✓ ✗

That's all.

///

We can rephrase the rules in the more formal language of "truth tables"

Rule 1: Every math. statement P has an opposite statement $\neg P$ (read "not P") with the opposite truth value.

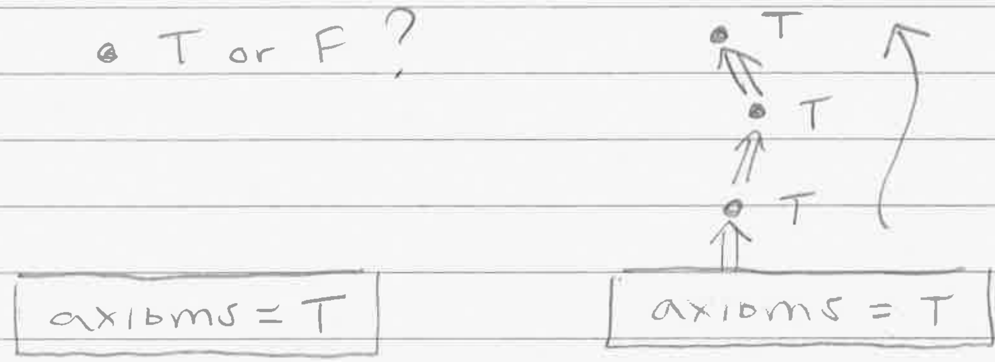
P	$\neg P$
T	F
F	T

Rule 2: The arrow $\Rightarrow$ is a function that sends an ordered pair of truth values to a truth value as follows

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

"Only $T \Rightarrow F$ is false because the T isn't flowing properly."

This explains my earlier schematic diagram of a proof:



"The axioms are the source of T. To prove a mathematical statement we drill down until we hit the axioms; then the T flows up."

But the axioms are also the source of F. We get an interesting duality by putting Rules 1 & 2 together.

☆ Logical Principle ("the contrapositive")

F flows <u>backwards</u>. In other words, the statements $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are <u>logically equivalent</u>.

We don't need to take this as a Rule because we can "prove" it.

"Proof": We can combine the truth tables from Rules 1 & 2 to get

P	Q	$\neg Q$	$\neg P$	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

"Only $T \Rightarrow F$ is false." Since the last two columns are the same we see that $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ always have the same truth value. In other words, they are

logically equivalent.

[Remark: This is much easier than trying to justify the contrapositive using verbal reasoning, right?]

9/14/15

I'll post HW2 later today.
It will be due Wed Sept 23.
Exam 1 is Fri Sept 25 in class.

Last time I used the method of contradiction to prove that $\sqrt{2}$ is "irrational", i.e., not a ratio of whole numbers

Then I stated the rules of logic we will use in this class

Recall:

Rule 1 ("excluded middle").

A mathematical statement is either T or F (not both, not neither). In other words, if P is a math statement then it has an opposite statement $\neg P$ defined by

P	$\neg P$
T	F
F	T

Rule 2 ("material implication").

"T flows along arrows"

In other words, for all math. statements P & Q we have

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

"only $T \Rightarrow F$ is false because the T isn't flowing properly."

From these two rules we derived the following.

☆ Logical Principle ("the contrapositive").

"F flows backwards".

In other words, for all math statements P & Q the statements

$$P \Rightarrow Q \quad \& \quad \neg Q \Rightarrow \neg P$$

are logically equivalent.

"Proof":

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The 3rd and 6th columns are equal. ///

Now let me explain how we use the contrapositive in mathematics. Its main application is the method of proof by contradiction.

Here's how it works:

Suppose we want to prove statement P. If we can build a sequence of arrows from $\neg P$ to a false statement

$$\neg P \Rightarrow Q_1 \Rightarrow Q_2 \Rightarrow \cdots \Rightarrow Q_k = F$$

then the F will flow backwards and tell us that $\neg P = F$, hence $P = T$.

In practice this means that we start by assuming $\neg P$ and show that this logically leads to a contradiction.

This is exactly what we did when we proved that $\sqrt{2}$ is irrational.

Here's a schematic diagram of the proof:

Let P = "$\sqrt{2}$ is rational",

$\neg P$ = "$\sqrt{2}$ is not rational".

we showed that

$\neg P$

$\Downarrow$

$\sqrt{2} = a/b$ for some whole numbers a & b with no common factor

$\Downarrow$

$a = 2k$ for some whole number k

$\Downarrow$

$b = 2l$ for some whole number l

$\Downarrow$

a & b have common factor 2

F

We conclude that $\neg P = F$, hence $P = T$.

[Remark: We say that this proof is <u>indirect</u> because it doesn't say anything about what $\sqrt{2}$ <u>is</u>; only what it is <u>not</u>.

To say what $\sqrt{2}$ <u>is</u> (e.g. $\sqrt{2} = 1.41421\cdots$) would require some ideas from the mathematical subject of <u>analysis</u>. (see MTH 433, 533/534).]

Now let's practice our skills by trying to prove that $\sqrt{3}$ is irrational.

[I won't write "Proof:" yet because we're just doing rough work at this point.]

<u>Assume</u> for contradiction that $\sqrt{3}$ <u>is</u> rational. Then we can write $\sqrt{3} = a/b$ where a & b are integers (i.e. "whole numbers") with no common factor. Square both sides to get

$$3 = a^2/b^2$$

$$\Longrightarrow 3b^2 = a^2.$$

Now what? If a^2 is a multiple of 3 then what does this tell us about a? Is **a** also a multiple of 3? <u>If</u> so, how could we prove it?

We want to prove that

a^2 is multiple of 3 $\Longrightarrow$ a is multiple of 3.

Maybe it will be easier to prove the (Logically equivalent) contrapositive statement

a not multiple of 3 $\Longrightarrow$ a^2 not multiple of 3.

So assume that a is not a multiple of 3.

Wait, it's hard to begin a proof with a negative statement. We need to turn this into a positive statement

"If a is not a multiple of 3, then

$a = \cdots$ "

Actually there are two separate ways for the number a to be not a mutiple of 3.

Case 1 : $a = 3k+1$ for some integer k.
Case 2 : $a = 3k+2$ for some integer k.

In case 1 we have

$$a^2 = (3k+1)^2$$
$$= 9k^2 + 6k + 1$$
$$= 3(3k^2 + 2k) + 1,$$

which is <u>not</u> a multiple of 3 (it has remainder 1 when divided by 3).

In case 2 we have

$$a^2 = (3k+2)^2$$
$$= 9k^2 + 12k + 4$$
$$= 3(3k^2 + 4k + 1) + 1,$$

which is also <u>not</u> a multiple of 3.

Putting both cases together gives

a <u>not</u> multiple of 3 $\Rightarrow$ a^2 <u>not</u> multiple of 3

hence

a^2 is multiple of 3 $\Rightarrow$ a <u>is</u> multiple of 3.

Back to the proof: we had

$$3b^2 = a^2.$$

Thus a^2 is a multiple of 3 and hence a is a multiple of 3, say $a = 3k$. We can substitute to get

$$3b^2 = (3k)^2$$
$$3b^2 = 9k^2$$
$$b^2 = 3k^2.$$

Thus b^2 is a multiple of 3, hence so is b. Say $b = 3l$.

Now we have $a = 3k$ & $b = 3l$. But this contradicts the fact that a and b have no common factors (except ± 1).

This completes the rough work. Now we're ready to go back and write the proof nicely ...

[but not today.]

9/16/15

HW 2 due Wed Sept 23.
Exam 1 Fri Sept 25 in class.
[I'll provide practice exams next week.]

Last time we did the rough work to show that $\sqrt{3}$ is irrational. Now we'll write a polished proof.

But first, let me introduce some convenient notation.

Notation:

- If S is a <u>set</u> (i.e. a collection of things) we write "$x \in S$" to mean that x is one of the things in the collection. We say

"$x \in S$" = "x is a member (or an element) of S".

[I guess "$\in$" stands for "element"...]

• Our favorite sets are sets of numbers:

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

is the set of natural numbers.

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

is the set of integers. ["Z" is for "Zahlen", i.e., "numbers".]

$\mathbb{Q}$ is the set of fractions of integers. We call this the set of rational numbers.

["rational" is for "ratio"; "Q" is for "quotient".]

$\mathbb{R}$ is the set of real numbers, i.e., numbers that have a decimal expansion.

[Note that $\sqrt{3} \in \mathbb{R}$. We want to show that $\sqrt{3} \notin \mathbb{Q}$.]

- If A and B are sets, we write "$A \subseteq B$" to mean that every element of A is also an element of B. We say

"$A \subseteq B$" = "A is a <u>subset</u> of B"

For example, we have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

- Suppose we want to say that <u>every</u> element of a set satisfies some given property.

Let S be a set and for each element $x \in S$ let $P(x)$ be some mathematical statement about x. Then we define the notations

"$\forall x \in S, P(x)$" = "The statement $P(x)$ holds for <u>all</u> elements $x \in S$"

$\int$

"$\exists x \in S, P(x)$" = "There exists an element $x \in S$ such that the statement $P(x)$ holds"

["$\forall$" is for "All"; "$\exists$" is for "Exists".]

For example, we have

"$A \subseteq B$" = "$\forall x \in A,\ x \in B$"
= "for all $x \in A$ we have $x \in B$".

- Given two integers $m, n \in \mathbb{Z}$ we will write

"$m \mid n$" = "$\exists k \in \mathbb{Z},\ n = mk$"
= "there exists an integer k such that $n = mk$".

In this case we say that "m divides n" or "n is divisible by m".

For practice, you should prove that

"$\forall n \in \mathbb{Z}, \ n \mid 0$"

"$\forall n \in \mathbb{Z}, \ 1 \mid n$"

are true statements.

Now I think we're ready to write a polished proof. First we will prove a lemma (i.e., a "little helper theorem").

Lemma: For all $n \in \mathbb{Z}$ we have

$$3 \mid n^2 \implies 3 \mid n.$$

Proof: We will prove the contrapositive statement

$$3 \nmid n \implies 3 \nmid n^2.$$

So assume that $3 \nmid n$. There are two ways this can happen:

Case 1: $\exists k \in \mathbb{Z}$ such that $n = 3k+1$.
In this case we have

$$\begin{aligned} n^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1, \end{aligned}$$

and hence $3 \nmid n^2$ [we'll prove this later; right now it's OK if it just seems true.].

Case 2: $\exists k \in \mathbb{Z}$ such that $n = 3k+2$.
In this case we have

$$\begin{aligned} n^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1, \end{aligned}$$

and hence $3 \nmid n^2$.

In either case we have shown that $3 \nmid n^2$, as desired.

///

Now for the main result.

Theorem: $\sqrt{3} \notin \mathbb{Q}$.

Proof: Assume for contradiction that $\sqrt{3} \in \mathbb{Q}$. Then $\exists\, a, b \in \mathbb{Z}$ such that

- $\sqrt{3} = a/b$
- $\nexists\, d > 1$ such that $d \mid a$ and $d \mid b$.

Square both sides to get

$$3 = a^2/b^2$$
$$3b^2 = a^2.$$

Since $3 \mid a^2$ the lemma implies $3 \mid a$, say $a = 3k$ with $k \in \mathbb{Z}$. Now substitute to get

$$3b^2 = a^2$$
$$3b^2 = (3k)^2$$
$$3b^2 = 9k^2$$
$$b^2 = 3k^2.$$

$$\lightning$$

Since $3 \mid b^2$, the Lemma implies that $3 \mid b$, say $b = 3l$ where $l \in \mathbb{Z}$.

But now we have $3 \mid a$ and $3 \mid b$, which contradicts the fact that a & b have no common divisor greater than 1.

We conclude that our original assumption, that $\sqrt{3} \in \mathbb{Q}$, is false.

QED.

9/18/15

HW 2 due next Wed Sept 23.
[Please note that I fixed a mistake in Problem 1 and changed the wording of Problem 2.]
Exam 1 is next Fri Sept 25 in class.

Last time we gave a polished proof that $\sqrt{3} \notin \mathbb{Q}$. On HW 2 you will give a similar proof that $\sqrt{5} \notin \mathbb{Q}$.

In fact, the following more general statement is true.

☆ Theorem: Let d be an integer. Then

$$\sqrt{d} \notin \mathbb{Z} \Longrightarrow \sqrt{d} \notin \mathbb{Q}.$$

That is, if d is not the square of an an integer then its square root is irrational. ///

Unfortunately, we don't have the technology to prove this yet.

In particular, I have not yet told you the formal definition (i.e. the axioms) of the set $\mathbb{Z}$. I will do this soon but first we need a bit more logical technology.

So far we have learned two "logical functions" $\neg$ & $\Rightarrow$ defined by the truth tables

P	$\neg P$
T	F
F	T

&

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

These functions are all we really need, but it is <u>convenient</u> to define two more auxiliary functions called

$$\vee \quad \& \quad \wedge .$$

They are defined by the truth tables

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

&

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

The technical names are "logical disjunction" ($\vee$) and "logical conjunction" ($\wedge$), but we usually just say

"$P \vee R$" = "P or R"
"$P \wedge R$" = "P and R".

Does that make any sense to you? Here's the reasoning:

- "P or Q" = T means that at <u>least</u> <u>one</u> of P or Q is true.
- "P and Q" = T means that <u>both</u> P and Q are true.

This can be generalized to define the disjunction and conjunction of any family of statements.

Let I be an index set and for each index $i \in I$ consider a statement P_i.

Then we define the disjunction

$$\text{“}\bigvee_{i\in I} P_i\text{”} := \text{“}\exists\, i \in I,\ P_i\text{”}$$
$$= \text{“There } \underline{\text{exists}} \text{ an index } i \text{ such that } P_i \text{ holds.”}$$

and the conjunction

$$\text{“}\bigwedge_{i\in I} P_i\text{”} := \text{“}\forall\, i \in I,\ P_i\text{”}$$
$$= \text{“The statement } P_i \text{ holds for } \underline{\text{all}} \text{ indices } i\text{”}.$$

[Does this agree with the definitions for two statements?

$$\text{“}P_1 \vee P_2\text{”} = \text{“There exists } \underline{\text{some}}\ i \in \{1,2\} \text{ such that } P_i \text{ holds”}$$

$\Updownarrow$

"$P_1 \wedge P_2$" = "The statement P_i holds for <u>all</u> $i \in \{1,2\}$".

Yes. It agrees.]

We saw how $\neg$ interacts with $\Longrightarrow$ (via the "contrapositive"). How does $\neg$ interact with $\vee$ and $\wedge$?

We need to find the <u>opposite</u> of the statement

"The statement P_i holds for <u>all</u> indices i."

... After some thinking, we believe that the opposite statement is

"There <u>exists</u> some index i such that the statement P_i does <u>not</u> hold."

Ok, so how do we say this in symbols?

we have

$$\neg(\forall i \in I, P_i) = \exists i \in I, \neg P_i$$

In other words, we have

(*) $$\neg\left(\bigwedge_{i\in I} P_i\right) = \bigvee_{i \in I} (\neg P_i)$$

Taking $\neg$ of both sides gives

$$\bigwedge_{i\in I} P_i = \neg\left(\bigvee_{i\in I}(\neg P_i)\right)$$

and then substituting $Q_i = \neg P_i$ (hence $P_i = \neg Q_i$) gives

(**) $$\bigwedge_{i\in I}(\neg Q_i) = \neg\left(\bigvee_{i\in I} Q_i\right)$$

The statements (*) and (**) are called de Morgan's Laws.

For poseterity, let's write (*) and (**) down in the case of two statements.

☆ Logical Principle ("de Morgan's laws"):

For all statements P and Q we have

- $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$
- $\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$.

[You will prove one of these on HW2.1 using a truth table.]

OK, so what? These logical principles are often helpful in proving mathematical theorems.

Example: Let $m, n \in \mathbb{Z}$. Prove that

"mn is odd" $\Longrightarrow$ "m is odd or n is odd".

Let P = "mn is odd"
Q = "m is odd"
R = "n is odd".

We want to prove $P \Longrightarrow (Q \vee R)$.

How?

Instead we will prove the contrapositive

$$\neg(Q \vee R) \Longrightarrow \neg P$$

Using de Morgan's law, this is the same as

$$(\neg Q \wedge \neg R) \Longrightarrow \neg P$$

In other words,

"m and n are both even" $\Longrightarrow$ "mn is even".

Proof: Suppose m and n are both even, say $m = 2k$ and $n = 2\ell$ for some $k, \ell \in \mathbb{Z}$. Then we have

$$\begin{aligned} mn &= (2k)(2\ell) \\ &= 2(2k\ell), \end{aligned}$$

which is even.

QED.

9/21/15

HW2 is due on Wed.
Exam 1 is Friday in class.
[I have handed out example exams from Fall 2012 and Fall 2013.]

We have now seen all of the logic we will need in this course. Let me summarize.

Every logical statement is either T or F (not both, not neither). There are four basic logical functions

$$\neg, \quad \Rightarrow, \quad \vee, \quad \wedge$$

defined by their truth tables:

P	$\neg P$
T	F
F	T

P	Q	$P \Rightarrow Q$	$P \vee Q$	$P \wedge Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	F
F	F	T	F	F

The functions $\vee$ & $\wedge$ can also be expressed in terms of logical quantifiers.

Given an "index set" I and a statement P_i for each index $i \in I$, we define

$$\text{“} \bigvee_{i \in I} P_i \text{”} = \text{“} \exists i \in I, P_i \text{”}$$

= "There exists an index $i \in I$ such that statement P_i holds."

$$\text{“} \bigwedge_{i \in I} P_i \text{”} = \text{“} \forall i \in I, P_i \text{”}$$

= "The statement P_i holds for all indices $i \in I$."

["$\forall$" is for "All"; "$\exists$" is for "Exists".]

Then we have two logical principles showing how $\neg$ interacts with the other logical functions.

☆ The Contrapositive: For all statements P & Q we have

$$\text{"} P \Rightarrow Q \text{"} = \text{"} \neg Q \Rightarrow \neg P \text{"}.$$

☆ De Morgan's Laws: For all statements P & Q we have

- $\text{"} \neg(P \vee Q) \text{"} = \text{"} \neg P \wedge \neg Q \text{"}$
- $\text{"} \neg(P \wedge Q) \text{"} = \text{"} \neg P \vee \neg Q \text{"}$.

We can also phrase de Morgan's Laws in terms of quantifiers:

- $\text{"} \neg(\exists i \in I, P_i) \text{"} = \text{"} \forall i \in I, \neg P_i \text{"}$
- $\text{"} \neg(\forall i \in I, P_i) \text{"} = \text{"} \exists i \in I, \neg P_i \text{"}$.

$///$

And that's all the logic we will ever need. Let's practice applying it to mathematical statements.

Let $a, b, m, n \in \mathbb{Z}$. Prove that

$$\text{"} a+b > m+n \text{"} \Longrightarrow \text{"} a>m \text{ or } b>n \text{"}.$$

///

Let's analyze the logic first. Define the statements

$P =$ "$a+b > m+n$"
$Q =$ "$a > m$"
$R =$ "$b > n$".

We want to prove that

$$P \Rightarrow (Q \vee R).$$

It is difficult to prove this kind of statement directly so first we will rewrite it in a more convenient form.

Let's practice our truth tables by proving that

$$\text{"} P \Rightarrow (Q \vee R) \text{"} = \text{"} (P \wedge \neg Q) \Rightarrow R \text{"}.$$

This is a "Boolean function in 3 variables" so our truth table will have $8 = 2^3$ rows.

P	Q	R	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$\neg Q$	$P \wedge \neg Q$	$(P \wedge \neg Q) \Rightarrow R$
T	T	T	T	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	F	T
F	F	F	F	T	T	F	T

Note that the 5th and 8th columns represent the same "Boolean function".

Thus we can rewrite $P \Rightarrow (Q \vee R)$ as

$$(P \wedge \neg Q) \Rightarrow R.$$

"$a+b > m+n$ and $a \leq m$" $\Rightarrow$ "$b > n$".

Can we prove it now?

Proof: Assume that we have $a+b > m+n$ and $a \leq m$. In this case we will show that $b > n$.

We will use the method of contradiction. So assume for contradiction that $b \leq n$. Adding a to both sides gives

(1) $$a+b \leq a+n.$$

On the other hand, adding n to both sides of the inequality $a \leq m$ (which is true by assumption) gives

(2) $$a+n \leq m+n.$$

Combining inequalities (1) & (2) then gives

$$a+b \leq m+n,$$

which is a contradiction.

QED.

Note: We could also have rewritten

$$P \Rightarrow (Q \vee R)$$

via the contrapositive and de Morgan's law as

$$(\neg Q \wedge \neg R) \Rightarrow \neg P \text{ , i.e.,}$$

$$\text{"}a \leq m \text{ and } b \leq n\text{"} \Rightarrow \text{"}a+b \leq m+n\text{"},$$

an then proved it this way. Maybe this would have been quicker, but doing it the other way was

Good Practice.

9/23/15

HW 2 due now.
Exam 1 in class Friday.

For review please see the provided practice exams and solutions.

Today I will do a semi-review.

Let A & B be sets. Recall that

$$\text{“}A \subseteq B\text{”} = \text{“}\forall x \in A,\ x \in B\text{”}.$$

In this case we say that A is a subset of B.

Q: What does it mean to say that A is not a subset of B?

A: $$\text{“}A \nsubseteq B\text{”} = \neg\,\text{“}A \subseteq B\text{”}$$

$$= \neg\,\text{“}\forall x \in A,\ x \in B\text{”}$$

$$= \text{“}\exists x \in A,\ x \notin B\text{”}$$

[“there exists an element of A that is not in B”.]

Now let $\mathcal{U}$ be some "universal set" [containing everything we might want to talk about], and let $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{U}$ be subsets of $\mathcal{U}$.

In this case there is another way to say "$A \subseteq B$":

$$\text{“}A \subseteq B\text{”} = \text{“}\forall x \in \mathcal{U},\ x \in A \Rightarrow x \in B\text{”}$$

Then computing the negation gives

$$\begin{aligned} \text{“}A \not\subseteq B\text{”} &= \neg\text{“}A \subseteq B\text{”} \\ &= \neg\text{“}\forall x \in \mathcal{U},\ x \in A \Rightarrow x \in B\text{”} \\ &= \text{“}\exists x \in \mathcal{U},\ x \in A \not\Rightarrow x \in B\text{”}. \end{aligned}$$

But what the heck does $\not\Rightarrow$ mean ?!

Problem 1 from Exam 1 of Fall 2012 used a truth table to show that for all statements P & Q we have

$$\text{“}P \Rightarrow Q\text{”} = \text{“}\neg P \vee Q\text{”}.$$

Then we can apply de Morgan's law to compute the negation:

$$\begin{aligned}(P \not\Rightarrow Q) &= \neg(P \Rightarrow Q)\\ &= \neg(\neg P \vee Q)\\ &= (\neg\neg P) \wedge (\neg Q)\\ &= P \wedge \neg Q.\end{aligned}$$

OK, whatever...
Let's apply this to analyze the statement "$A \not\subseteq B$".

$$\begin{aligned}(A \not\subseteq B) &= \neg(A \subseteq B)\\ &= \neg(\forall x \in \mathcal{U},\ x \in A \Rightarrow x \in B)\\ &= (\exists x \in \mathcal{U},\ x \in A \not\Rightarrow x \in B)\\ &= (\exists x \in \mathcal{U},\ x \in A \wedge x \notin B).\end{aligned}$$

[Here we used $P = (x \in A)$ & $Q = (x \in B)$ so that $(P \not\Rightarrow Q) = (P \wedge \neg Q)$.].

In other words, "$A \not\subseteq B$" means that there exists a thing x in the universe such that x is in A but not in B.

Does that make sense?

You might think that this is all hopelessly pedantic, but I'll show you that it really does come up in mathematics.

The fundamental concept of Calculus is the "limit". Given a function $f : \mathbb{R} \to \mathbb{R}$ of one real variable, we write

$$\lim_{x \to a} f(x) = L$$

and we say that "the limit of $f(x)$ as x approaches a is L". But what does this really mean?

The modern definition of "limit" was developed by Augustin-Louis Cauchy in the 1820s.

His definition is horrifying, but it is still the standard one.

Here it is.

Definition: Let $f : \mathbb{R} \to \mathbb{R}$. We will write

$$\text{“} \lim_{x \to a} f(x) = L \text{”}$$

to mean that

$$\text{“} \forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x-a| < \delta \implies |f(x) - L| < \varepsilon \text{”}.$$

//

Yes, I'm serious. We can phrase it in terms familiar to us if we define the sets

$$\mathbb{R}_{>0} = \{ x \in \mathbb{R} : x > 0 \}$$
$$= \text{the set of positive real numbers}$$

$$B_r(p) := \{ x \in \mathbb{R} : |x - p| < r \}$$
$$= \text{the ball (line segment in this case) of radius } r \text{ centered at } p.$$

Then the definition becomes

" $\lim_{x \to a} f(x) = L$ "

$=$ " $\forall \varepsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}, \forall x \in \mathbb{R}, x \in B_\delta(a) \Rightarrow f(x) \in B_\varepsilon(L)$ "

///

It's hard enough to use this definition to prove that $\lim_{x \to a} f(x) = L$.

[Example: Prove that $\lim_{x \to 1} (3x+1) = 4$.

Proof: Let $\varepsilon > 0$ be given. Then we can choose some $\delta > 0$, in fact we will choose $\delta = \varepsilon/3$, such that for all $x \in \mathbb{R}$ we have

$$
\begin{aligned}
|x-1| < \delta &\Longrightarrow |x-1| < \varepsilon/3 \\
&\Longrightarrow 3|x-1| < \varepsilon \\
&\Longrightarrow |3(x-1)| < \varepsilon \\
&\Longrightarrow |3x-3| < \varepsilon \\
&\Longrightarrow |(3x+1)-4| < \varepsilon,
\end{aligned}
$$

as desired. ///]

But what if you were asked to prove

$$\neg\left(\lim_{x \to a} f(x) = L\right) \ ?$$

As an undergrad, I was asked to prove this but my logic skills were not up to the task. If I had known then what you know now, I could have just applied the rules (de Morgan's Law) to get

$$\neg\left(\lim_{x \to a} f(x) = L\right)$$

$$= \neg\left(\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon\right)$$

$$= \left(\exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}, |x-a| < \delta \not\Rightarrow |f(x) - L| < \varepsilon\right)$$

$$= \left(\exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}, |x-a| < \delta \text{ and } |f(x) - L| \geq \varepsilon\right).$$

At least then I would have known what to prove.

P.S. The stuff about limits is not on the exam.