Math 230 E **Fall 2013**
Exam 1 — Fri Oct 4 **Drew Armstrong**

There are 4 problems, worth 5 points each. This is a closed book test. Anyone caught cheating will receive a score of **zero**.

Problem 1. Let $a, b, q, r \in \mathbb{Z}$ be integers such that $a = qb + r$, and consider any $d \in \mathbb{Z}$.

(a) State the definition of the symbol "$d|a$".

There exists $a' \in \mathbb{Z}$ such that $a = da'$.

(b) Prove that "($d|a$ AND $d|b$) $\Rightarrow$ $d|r$".

Proof. Assume that $d|a$ and $d|b$, that is, there exist $a', b' \in \mathbb{Z}$ such that $a = da'$ and $b = db'$. Then we have

$$r = a - qb = da' - qdb' = d(a' - qb'),$$

hende $d|r$. □

(c) Prove that "($d|b$ AND $d|r$) $\Rightarrow$ $d|a$".

Proof. Assume that $d|b$ and $d|r$, that is, there exist $b', r' \in \mathbb{Z}$ such that $b = db'$ and $r = dr'$. Then we have

$$a = qb + r = qdb' + dr' = d(qb' + r'),$$

hence $d|a$. □

Problem 2.

(a) Use a truth table to prove that "NOT (P OR Q)" $\equiv$ "(NOT P) AND (NOT Q)".

Proof.

P	Q	P OR Q	NOT (P OR Q)	NOT P	NOT Q	(NOT P) AND (NOT Q)
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

□

(b) Let $m, n \in \mathbb{Z}$. What is the **contrapositive** of the following statement?

"($m + n$ is odd) $\Rightarrow$ (m is odd) OR (n is odd)"

By part (a), the contrapositive statement is

"(m is even) AND (n is even) $\Rightarrow$ ($m + n$ is even)"

(c) Prove the statement from part (b).

Proof. Assume that m is even and n is even, that is, there exist $m', n' \in \mathbb{Z}$ such that $m = 2m'$ and $n = 2n'$. Then we have have

$$m + n = 2m' + 2n' = 2(m' + n'),$$

hence $m + n$ is even. □

Problem 3.

(a) Accurately state the Division Algorithm (Theorem).

For all $a, b \in \mathbb{Z}$ with $b \neq 0$, there exist unique $q, r \in \mathbb{Z}$ with the properties
- $a = qb + r$,
- $0 \leq r < |b|$.

(b) We say that an integer $n \in \mathbb{Z}$ is **even** if there exists $q \in \mathbb{Z}$ such that $n = q2$. Use the Division Algorithm to **prove** that 3 is not even. [Hint: Assume for contradiction that 3 is even. Divide 3 by 2 and think about the remainder.]

Proof. **Assume for contradiction** that 3 is even, that is, there exists $q \in \mathbb{Z}$ such that $3 = q2$. Then we have $3 = q2+0$ with $0 \leq 0 < |2|$. But we also have $3 = 1\cdot 2+1$ with $0 \leq 1 < |2|$. By the uniqueness of the remainder (Division Algorithm), we conclude that $0 = 1$. This **contradiction** proves that 3 is not even. $\square$

Problem 4. In this problem (and only in this problem) you may use the notation $\frac{a}{b}$. You may also assume that for all $n \in \mathbb{Z}$ the statement "$3|n^2 \Rightarrow 3|n$" is true.

(a) Prove that if $\sqrt{3}$ is not a fraction, then $\sqrt{12}$ is not a fraction. [Hint: $\sqrt{12} = 2\sqrt{3}$.]

Proof. We will prove the contrapositive statement. So assume that $\sqrt{12}$ is a fraction, say $\sqrt{12} = \frac{a}{b}$. Then we have

$$2\sqrt{3} = \sqrt{12} = \frac{a}{b} \Rightarrow \sqrt{3} = \frac{a}{2b},$$

hence $\sqrt{3}$ is a fraction. $\square$

(b) Let Q be the statement "$\sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with no common factor". Use the method of contradiction to prove that Q is false. [Hint: Assume that Q is true.]

Proof. **Assume for contradiction** that we can write $\sqrt{3} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ have no common factor. Multiplying both sides by b gives $a = b\sqrt{3}$ and squaring gives $a^2 = 3b^2$. Since $3|a^2$ we have $3|a$, say $a = 3k$. But then $3b^2 = a^2 = 9k^2$, hence $b^2 = 3k^3$. We conclude that $3|b^2$ and hence $b|3$. Thus a and b are both divisible by 3, which **contradicts** the fact that they have no common divisor. Hence our original assumption is false. $\square$

(c) Let $P =$"$\sqrt{3}$ is a fraction". You may assume that $P \Rightarrow Q$ is true. Now put parts (a) and (b) together to prove that $\sqrt{12}$ is not a fraction.

Proof. We proved in part (b) that NOT Q is true. By the contrapositive, this proves that NOT P is true, that is, $\sqrt{3}$ is not a fraction. By part (a) this implies that $\sqrt{12}$ is not a fraction. $\square$