Math 230 D **Fall 2012**
Homework 5 **Drew Armstrong**

Problem 1. Recall that $a \equiv b \bmod n$ means that $n|(a-b)$. Use induction to prove that for all $n \geq 2$, the following holds:

"**if** $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ such that each $a_i \equiv 1 \bmod 4$, **then** $a_1 a_2 \cdots a_n \equiv 1 \bmod 4$."

[Hint: Call the statement $P(n)$. Note that $P(n)$ is a statement about **all** collections of n inegers. Therefore, when proving $P(k) \Rightarrow P(k+1)$ you must say "Assume that $P(k) = T$ and consider any $a_1, a_2, \ldots, a_{k+1} \in \mathbb{Z}$." What is the base case?]

Problem 2. Use induction to prove that for all integers $n \geq 2$ the following statement holds: "**If** p is prime and $p|a_1 a_2 \cdots a_n$ for some integers $a_1, a_2, \ldots, a_n \geq 2$, **then** there exists i such that $p|a_i$." [Hint: Call the statement $P(n)$. Use Euclid's Lemma for the induction step. You don't need to prove it again. In fact, there's no new math in this problem; just setting up notation and not getting confused.]

Problem 3. Use induction to prove that for all integers $n \geq 1$ we have

$$\text{“}1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.\text{”}$$

This result appears in the *Aryabhatiya* of Aryabhata (499 CE, when he was 23 years old). [Hint: You may assume the result $1 + 2 + \cdots + n = n(n+1)/2$.]

Problem 4. Consider the following two statements/principles.

PSI: **If** $P : \mathbb{N} \to \{T, F\}$ is a family of statements satisfying

- $P(1) = T$, and
- for any $k \geq 1$ we have $[P(1) = P(2) = \cdots = P(k) = T] \Rightarrow [P(k+1) = T]$.

then $P(n) = T$ for all $n \in \mathbb{N}$.

WO: Every nonempty subset $K \subseteq \mathbb{N} = \{1, 2, 3, \ldots\}$ has a least element.

Now **Prove** that PSI $\Rightarrow$ WO. [Hint: Assume PSI and show that the (equivalent) contrapositive of WO holds; i.e., that if $K \subseteq \mathbb{N}$ has **no** least element then $K = \emptyset$. To do this you can use PSI to show that the complement K^c is all of $\mathbb{N}$. Let $P(n)$ be the statement "$n \in K^c$" and show using PSI that $P(n) = T$ for all $n \in \mathbb{N}$.]

Problem 5. Let $d(n)$ be the number of binary strings of length n that contain no consecutive 1's. For example, there are 5 such strings of length 3:

000, 100, 010, 001, 101.

Hence $d(3) = 5$. Prove that $d(n)$ are (essentially) the Fibonacci numbers, and hence give a closed formula for $d(n)$. [Hint: First show that $d(n) = d(n-1) + d(n-2)$ for all $n \geq 3$. [Hint: The first digit (actually, bit) of a string can be either 1 or 0.] Then use PSI.]