Math 230 D — Fall 2012
Homework 2 — Drew Armstrong

Problem 1.

(a) Prove that the product of two odd numbers is odd.

(b) Prove that 3^n is odd for all integers $n \geq 1$. [It's easy to prove that, say, 3^{101} is odd. But how will you prove it for infinitely many different n without having to say infinitely many things? I'll warn you: There is a big issue here called **induction**.]

(c) Assume for the moment that there exists a number x such that $2^x = 3$ and call it $x = \log_2(3)$. Prove that $\log_2(3)$ is not a fraction.

Proof of (a). Let m and n be two odd integers, say $m = 2k+1$ and $n = 2\ell+1$. It follows that the product $mn = (2k+1)(2\ell+1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1$ is odd. $\square$

[There is an issue lurking in Problem 1(b). I'll give two solutions. The first solution is acceptable for now, but is not really correct. The second solution is correct, but I don't expect that you would come up with this by yourself. (We will discuss the issue of "induction" in due time.)]

Proof of (b). **Solution 1.** In part (a) we showed that the product of two odd integers is odd. Since 3 is odd we conclude that $3^2 = 3 \cdot 3$ odd. Then since 3^2 is odd we conclude that $3^3 = 3^2 \cdot 3$ is odd. Continuing in this way we conclude that 3^n is odd for all integers $n \geq 1$. [The issue with this solution is the words "continuing in this way"; what does that even mean?] **Solution 2.** Suppose for contradiction that there exists some integer $m \geq 1$ such that 3^m is **even**, and suppose that m is the **smallest** integer with this property. Since 3^1 is certainly odd we know that $m > 1$ and hence 3^{m-1} is **odd**. Finally, by part (a) we conclude that $3^m = 3 \cdot 3^{m-1}$ is odd. This **contradiction** means that there does **not** exist an integer $m \geq 1$ such that 3^m is even. $\square$

[For the next proof, it's okay if you assume that $0 < x$. I kind of thought you would.]

Proof of (c). Suppose for contradiction that $x = \log_2(3) = m/n$ for some integers m and n. Since $x > 0$ — why is this? — we can assume that $m \geq 1$ and $n \geq 1$. By definition this means that $2^{m/n} = 3$, and raising both sides of this equation to the power n gives $2^m = 3^n$. But $2^m = 2(2^{m-1})$ is even and we proved in part (b) that 3^n is odd. Contradiction. $\square$

[But if you don't want to assume $0 < x$, then I appreciate your skepticism. In this case, the proof might go as follows.]

More Rigorous Proof of (c). Suppose for contradiction that $x = \log_2(3) = m/n$ for some integers m and n. Without loss of generality, we can assume that $n > 0$ and absorb any negative sign into the numerator. Now there are three cases. **Case 1:** If $m = 0$ then we have $2^0 = 3$, which is false. **Case 2:** If $m \geq 1$ then we have $2^{m/n} = 3$, and raising both sides to the power n gives $2^m = 3^n$. But $2^m = 2(2^{m-1})$ is even and by part (a) we know that 3^n is odd. Contradiction. **Case 3:** If $m \leq -1$, let $m' = -m \geq 1$. Then raising both sides of $2^{m/n} = 3$ to the power n gives $2^m = 3^n$, which is the same as $1/2^{m'} = 3^n$. Then we multiply both sides by $2^{m'}$ to get $1 = 3^n 2^{m'}$ with $n \geq 1$ and $m' \geq 1$. This is a contradiction because $3^n \geq 3$ and $2^{m'} \geq 2$. (Why is that true? Stay tuned.)

In any case, we have a contradiction. $\square$

[Note: It is not possible for us to give a completely rigorous proof right now, because I have not told you the axioms of the integers. In fact, even after I tell you the axioms, we won't want to reduce every proof to the axioms. I think the second proof of (c) above is already too rigorous.]

Problem 2. Prove that there is no perfect square of the form $4k+3$. That is, prove that there do not exist integers n and k such that $n^2 = 4k+3$.

Proof. Let n be an integer. Using division with remainder we can write n as $4k+0$, $4k+1$, $4k+2$ or $4k+3$ for some integer k. We will compute n^2 in each case. **Case 0:** $n^2 = (4k+0)^2 = 16k^2 = 4(4k^2)+0$. **Case 1:** $n^2 = (4k+1)^2 = 16k^2+8k+1 = 4(4k^2+2k)+1$. **Case 2:** $n^2 = (4k+2)^2 = 16k^2+16k+4 = 4(4k^2+4k+1)+0$. **Case 3:** $n^2 = (4k+3)^2 = 16k^2+24k+9 = 4(4k^2+6k+2)+1$. In any case we observe that the remainder of n^2 when divided by 4 is **not** equal to 3. □

[If you don't like that, here's a more elegant proof suggested by a student.]

More elegant proof. Suppose for contradiction that there exist integers n and k such that $n^2 = 4k+3$. Since $n^2 = 4k+3 = 2(2k+1)+1$ is odd, it follows that n must be odd. (Certainly, if n even then so is n^2. Now take the contrapositive.) Thus we can write $n = 2\ell+1$ for some integer ℓ. Finally we have

$$\begin{aligned} n^2 &= 4k+3 \\ (2\ell+1)^2 &= 4k+3 \\ 4\ell^2+4\ell+1 &= 4k+3 \\ 4(\ell^2+\ell-k) &= 2 \\ 2(\ell^2+\ell-k) &= 1, \end{aligned}$$

which is a contradiction. □

Problem 3. Let P, Q and R be logical statements.

(a) Use a truth table to prove that the statement $P \Rightarrow (Q \text{ OR } R)$ is logically equivalent to the statement $(P \text{ AND NOT } Q) \Rightarrow R$.

(b) Use a truth table to prove that the statement $(P \text{ OR } Q) \Rightarrow R$ is logically equivalent to the statement $(P \Rightarrow R) \text{ AND } (Q \Rightarrow R)$.

Proof of (a). (What can I say?) Observe that the following truth table is correct, and that the fifth and eighth columns are equal.

P	Q	R	Q OR R	$P \Rightarrow (Q \text{ OR } R)$	NOT Q	P AND NOT Q	$(P \text{ AND NOT } Q) \Rightarrow R$
T	T	T	T	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	F	T
F	F	F	F	T	T	F	T

□

Proof of (b). Observe that the following truth table is correct, and that the fifth and eighth columns are equal.

P	Q	R	P OR Q	$(P$ OR $Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R)$ AND $(Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$\square$

Problem 4. Let m and n be integers. Prove that

$$\text{“}(m \text{ is even OR } n \text{ is even}) \Leftrightarrow mn \text{ is even”}.$$

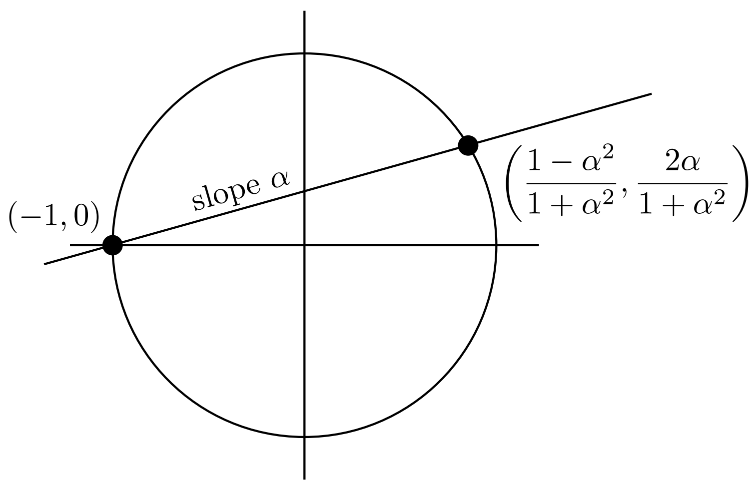
Explicitly state any logical principles that you use. [Hint: You will need Problem 3.]

Proof. Consider the statements $P =$“m is even”, $Q =$“n is even” and $R =$“mn is even”. We wish to prove that $(P$ OR $Q) \Rightarrow R$.

First we will show that $(P$ OR $Q) \Rightarrow R$, which by Problem 3(b) is equivalent to $(P \Rightarrow R)$ AND $(Q \Rightarrow R)$. To show $P \Rightarrow R$, suppose that m is even, say $m = 2k$. Then $mn = 2kn = 2(kn)$ is even as desired. Similarly, to show that $Q \Rightarrow R$, suppose that n is even, say $n = 2\ell$. Then $mn = m2\ell = 2(m\ell)$ is even.

Next we will show that $R \Rightarrow (P$ OR $Q)$, which by Problem (a) is equivalent to $(R$ AND NOT $P) \Rightarrow Q$. So suppose that R AND NOT P, i.e. that mn is even and m is odd. It follows that n is even since otherwise by Problem 1(a) we would have mn odd, which is a contradiction. $\square$

Problem 5. Draw a line of slope α through the point $(-1, 0)$ on the unit circle.



(a) Prove that the other point of intersection is $\left(\frac{1-\alpha^2}{1+\alpha^2}, \frac{2\alpha}{1+\alpha^2}\right)$.

(b) Prove that α is a fraction if and only if $\frac{1-\alpha^2}{1+\alpha^2}$ and $\frac{2\alpha}{1+\alpha^2}$ are **both** fractions.

Proof of (a). We follow Descartes by writing down the equation of the line $(y = \alpha(x + 1))$ and the equation of the circle $(x^2 + y^2 = 1)$. Then the points of intersection are the values of $(x, y) \in \mathbb{R}^2$ that satisfy both equations simultaneously. First we square the equation of the line to get $y^2 = \alpha^2(x+1)^2 = \alpha^2x^2 + 2\alpha^2x + \alpha^2$. Then we substitute this into the equation of the circle to get

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + (\alpha^2x^2 + 2\alpha^2x + \alpha^2 + 1) &= 1 \\ (1+\alpha^2)x^2 + 2\alpha^2x + (\alpha^2 - 1) &= 0. \end{aligned}$$

Next we apply the "quadratic formula" to get

$$\begin{aligned} x &= \frac{1}{2(1+\alpha^2)}[-2\alpha^2 \pm \sqrt{4\alpha^4 - 4(1+\alpha^2)(\alpha^2-1)}] \\ &= \frac{1}{2(1+\alpha^2)}[-2\alpha^2 \pm \sqrt{4\alpha^4 - 4(\alpha^4-1)}] \\ &= \frac{1}{2(1+\alpha^2)}[-2\alpha^2 \pm \sqrt{4}] \\ &= \frac{1}{2(1+\alpha^2)}[-2\alpha^2 \pm 2] \\ &= \frac{1-\alpha^2}{1+\alpha^2} \quad \text{or} \quad -1. \end{aligned}$$

Finally, we substitute these values of x back into the equation $y = \alpha(x+1)$ to get the solutions $(x,y) = (-1,0)$ and $\left(\frac{1-\alpha^2}{1+\alpha^2}, \frac{2\alpha}{1+\alpha^2}\right)$. □

Proof of (b). First suppose that α is a fraction. That is, suppose that $\alpha = a/b$ for some integers a and b, with $b \neq 0$. Then by multiplying the numerator and denominator by b^2 we get

$$\frac{1-\alpha^2}{1+\alpha^2} = \frac{1-a^2/b^2}{1+a^2/b^2} = \frac{b^2-a^2}{b^2+a^2}$$

and

$$\frac{2\alpha}{1+\alpha^2} = \frac{2a/b}{1+a^2/b^2} = \frac{2ab}{b^2+a^2},$$

which are both fractions of integers.

Conversely, suppose that $X := \frac{1-\alpha^2}{1+\alpha^2} = \frac{a}{b}$ and $Y := \frac{2\alpha}{1+\alpha^2} = \frac{c}{d}$ for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$. Then from the equation of the line ($y = \alpha(x+1)$) we get

$$\alpha = \frac{Y}{X+1} = \frac{c/d}{(a+b)/b} = \frac{bc}{d(a+b)},$$

which is a fraction of integers. □

[Note: This gives us a **bijection** between the "rational points" on the circle (except for $(-1,0)$) and the set of all "rational numbers". For any rational number $\alpha = a/b$ we get a rational point $\left(\frac{b^2-a^2}{b^2+a^2}, \frac{2ab}{b^2+a^2}\right)$ on the unit circle, and every rational point on the unit circle has this form. Of what possible use could this be?]