Math 230 D — Fall 2012
Exam 3 — Fri Nov 30 — Drew Armstrong

There are 4 problems, worth 5 points each. This is a closed book test. Anyone caught cheating will receive a score of **zero**.

Problem 1.

(a) Accurately state the Principle of Strong Induction.

Let $P(n)$ be a logical statement depending on an integer n. If

- $P(b) = T$, and
- For all integers $k \geq b$ we have

$$(P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)) \Rightarrow P(k+1),$$

then it follows that $P(n) = T$ for all $n \geq b$.

Now let WO=Well Ordering Principle, PI=Principle of Induction and PSI=Principle of Strong Induction. Evaluate the following statements as **True** or **False**.

(b) WO $\Rightarrow$ PI

Recall that WO, PI, and PSI are all logically equivalent. Hence WO $\Rightarrow$ PI is true.

(c) NOT WO $\Rightarrow$ NOT PSI

This is logically equivalent to PSI $\Rightarrow$ WO, which is true.

Problem 2. Prove that for all integers $n \geq 1$ the number $n^3 + 2n$ is divisible by 3.

(a) Define a logical statement $P(n)$.

$$P(n) := \text{“}3|(n^3 + 2n)\text{”}$$

(b) Verify that $P(1)$ is true.

Note that $P(1) =$ “3|3”, which is clearly true.

(c) Let $k \geq 1$ and assume that $P(k)$ is true. In this case, prove that $P(k+1)$ is also true.

We assume that $3|(k^3 + 2k)$, which means that $k^3 + 2k = 3\ell$ for some $\ell \in \mathbb{Z}$. Then we have

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 1 \\ &= (k^3 + 2k) + 3k^2 + 3k + 3 \\ &= 3\ell + 3(k^2 + k + 1) \\ &= 3(\ell + k^2 + k + 1),\end{aligned}$$

hence $P(k+1)$ is true.

Problem 3.
(a) Accurately state Pascal's Recurrence for binomial coefficients $\binom{n}{k}$.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(b) Prove Pascal's Recurrence using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Using the formula gives

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{(n-k)}{(n-k)} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{[k + (n-k)](n-1)!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

Problem 4. We say that a group of horses is *monochromatic* if each horse in the group has the same color. Consider the following statement:

$P(n) =$ "**Every** group of n horses is monochromatic."

(a) Let $k \geq 2$ and assume that $P(k)$ is true. In this case, prove that $P(k+1)$ is also true.

Proof. We have assumed that every group of k horses is monochromatic. Now consider any group of $k+1$ horses, say $h_1, h_2, \ldots, h_k, h_{k+1}$. By assumption the group

$$h_1, h_2, \ldots, h_k \tag{1}$$

is monochromatic, as well as the group

$$h_2, \ldots, h_k, h_{k+1}. \tag{2}$$

It remains only to check that h_1 and h_{k+1} have the same color. Since $k+1 \geq 3$ there exists some horse h_i with $1 < i < k+1$. By (1) we know that h_1 and h_i have the same color and by (2) we know that h_i and h_{k+1} have the same color. By transitivity we conclude that h_1 and h_{k+1} have the same color. We conclude that the group is monochromatic. □

(b) This does **not** imply that $P(n)$ is true for all $n \geq 2$. **Why not?** [Hint: There exist white horses. There exist black horses.] Answer: Because $P(2)$ is false.