Math 224 **Summer 2019**
Quiz 5 **Drew Armstrong**

Problem 1. Let $X_1, X_2, \ldots, X_{36}$ be an iid sample with $E[X_i] = 1/2$ and $\text{Var}(X_i) = 1/4$. Consider the sample mean:

$$\overline{X} = \frac{1}{36}(X_1 + X_2 + \cdots + X_{36}).$$

(a) Compute $E[\overline{X}]$ and $\text{Var}(\overline{X})$.

Since the expected value is linear we have

$$E[\overline{X}] = \frac{E[X_1] + E[X_2] + \cdots + E[X_{36}]}{36} = \frac{36 \cdot (1/2)}{36} = \boxed{1/2.}$$

Since the sample is independent we also have

$$\text{Var}(\overline{X}) = \frac{\text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_{36})}{36^2} = \frac{36 \cdot (1/4)}{36^2} = \boxed{1/144.}$$

(b) Use the Central Limit Theorem to estimate the probability $P(0.48 < \overline{X} < 0.54)$.

From the CLT we know that $(\overline{X} - 0.5)/\sqrt{1/144} = (\overline{X} - 0.5)/(1/12) = 12(\overline{X} - 0.5)$ is approximately standard normal. Therefore we have

$$\begin{aligned}
P(0.48 < \overline{X} < 0.52) &= P(-0.02 < \overline{X} - 0.5 < 0.04) \\
&= P(-0.24 < 12(\overline{X} - 0.5) < 0.48) \\
&\approx P(-0.24 < Z < 0.48) \\
&= \Phi(0.48) - \Phi(-0.24) \\
&= \Phi(0.48) + \Phi(0.24) - 1 \\
&= 0.6844 + 0.5948 - 1 = \boxed{27.92\%.}
\end{aligned}$$

(c) Use the Central Limit Theorem to find a number k such that $P(\overline{X} > k) \approx 5\%$.

Since $12(\overline{X} - 0.5)$ is approximately standard normal we will work backwards:

$$\begin{aligned}
P(Z > 1.645) &= 5\%, \\
P\left(12(\overline{X} - 0.5) > 1.645\right) &\approx 5\%, \\
P\left(\overline{X} > \frac{1.645}{12} + 0.5\right) &\approx 5\%, \\
P\left(\overline{X} > \boxed{0.64}\right) &\approx 5\%.
\end{aligned}$$

Problem 2. The following iid sample comes from a normal distribution $N(\mu, \sigma^2)$:

1.3	2.1	2.5	3.1

(a) Compute the sample mean $\overline{X}$ and sample variance S^2.

We have

$$\overline{X} = \frac{1.3 + 2.1 + 2.5 + 3.1}{4} = \frac{9.1}{4} = 2.275$$

and

$$S^2 = \frac{(1.3 - 2.25)^2 + (2.1 - 2.25)^2 + (2.5 - 2.25)^2 + (3.1 - 2.25)^2}{3} = 0.57.$$

(b) Compute a two-sided 98% confidence interval for μ.

At $(1 - \alpha)100\% = 98\%$ confidence we have $\alpha = 0.2$ and hence $t_{\alpha/2}(3) = 4.541$:

$$\begin{aligned} \overline{X} - t_{\alpha/2}(n-1) \cdot \sqrt{\tfrac{S^2}{n}} &< \mu < \overline{X} + t_{\alpha/2}(n-1) \cdot \sqrt{\tfrac{S^2}{n}} \\ 2.25 - 4.541 \cdot \sqrt{\tfrac{0.57}{4}} &< \mu < 2.25 - 4.541 \cdot \sqrt{\tfrac{0.57}{4}} \\ 2.25 - 1.71 &< \mu < 2.25 - 1.71. \end{aligned}$$

(c) Test the hypothesis $H_0 =$ "$\mu = 2$" against $H_1 =$ "$\mu > 2$" at 5% significance.

At significance $\alpha = 5\%$ we have $t_\alpha(3) = 2.353$. The rejection region is

$$\begin{aligned} \overline{X} &> \mu_0 + t_\alpha(n-1) \cdot \sqrt{\frac{S^2}{n}} \\ \overline{X} &> 2 + 2.353 \cdot \sqrt{\frac{0.57}{4}} \\ \overline{X} &> 2 + 0.89. \end{aligned}$$

Since $\overline{X} = 2.25$ we **do not reject** H_0. [That's good, because my computer generated this sample from $N(2, 1)$.]