Math 224 **Summer 2019**
Homework 3 **Drew Armstrong**

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1. An urn contains 3 red and 2 green balls. Your friend selects two balls (at random and without replacement) and tells you that at least one of the balls is red. What is the probability that the other ball is also red? [Hint: Let $A = \{\text{1st ball is red}\}$ and $B = \{\text{2nd ball is red}\}$. You want to compute the probability of $A \cap B$, assuming that $A \cup B$ happens.]

First Solution. Let the two balls be ordered so that $\#S = 5 \times 4 = 20$. Consider the events

$$\begin{aligned} A &= \{\text{1st ball is red}\}, \\ B &= \{\text{2nd ball is red}\}. \end{aligned}$$

Now observe that we have $P(A) = P(B) = 3/5$ and $P(B|A) = 2/4 = 1/2$ (if the first ball is red then 2 of the 4 remaining balls are red). It follows that

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{3}{5} \cdot \frac{1}{2} = \frac{3}{10}.$$

Alternatively, there are $3 \times 2 = 6$ ways to choose two red balls, so that

$$P(A \cap B) = \frac{\#(A \cap B)}{\#S} = \frac{3 \times 2}{5 \times 4} = \frac{6}{20} = \frac{3}{10}.$$

From this we can compute

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{5} + \frac{3}{5} - \frac{3}{10} = \frac{9}{10}.$$

Finally, we have

$$\begin{aligned} &P(\text{“both balls are red”}\,|\,\text{“at least one ball is red”}) \\ &= P(A \cap B | A \cup B) \\ &= \frac{P\left((A \cap B) \cap (A \cup B)\right)}{P(A \cup B)} \\ &= \frac{P(A \cap B)}{P(A \cup B)} \\ &= \frac{3/10}{9/10} = \frac{3}{9} = \frac{1}{3}. \end{aligned}$$

Second Solution. Let the two balls be unordered so that $\#S = \binom{5}{2} = 10$. Let X be the number of red balls we get, so that

$$P(X = k) = \frac{\binom{3}{k}\binom{2}{2-k}}{\binom{5}{2}}.$$

We are looking for the probability

$$\begin{aligned}
P(X = 2|X \geq 1) &= \frac{P(X = 2 \text{ and } X \geq 1)}{P(X \geq 1)} \\
&= \frac{P(X = 2)}{P(X \geq 1)} \\
&= \frac{P(X = 2)}{1 - P(X = 0)} \\
&= \frac{\binom{3}{2}\binom{2}{0}/\binom{5}{2}}{1 - \binom{3}{0}\binom{2}{2}/\binom{5}{2}} \\
&= \frac{3/10}{1 - 1/10} \\
&= \frac{3/10}{9/10} = \frac{3}{9} = \frac{1}{3}.
\end{aligned}$$

2. There are two bowls on a table. The first bowl contains 3 red chips and 2 green chips. The second bowl contains 3 red chips and 4 green chips. Your friend walks up to the table and chooses one chip at random. Consider the events

$$\begin{aligned}
B_1 &= \{\text{chip comes from 1st bowl}\}, \\
B_2 &= \{\text{chip comes from 2nd bowl}\}, \\
R &= \{\text{chip is red}\}.
\end{aligned}$$

(a) Compute the forwards probabilities $P(R|B_1)$ and $P(R|B_2)$.
(b) Assuming that your friend is equally likely to choose either bowl (i.e., $P(B_1) = P(B_2) = 1/2$), compute the probability $P(R)$ that the chip is red.
(c) Compute the backwards probability $P(B_1|R)$. That is, assuming that your friend chose a red chip, what is the probability that it came from the first bowl?

(a) We have $P(R|B_1) = \frac{3}{3+2} = \frac{3}{5}$ and $P(R|B_2) = \frac{3}{3+4} = \frac{3}{7}$.

(b) The Law of Total Probability gives

$$\begin{aligned}
R &= (R \cap B_1) \cup (R \cap B_2) \\
P(R) &= P(R \cap B_1) + P(R \cap B_2) \\
P(R) &= P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2) \\
&= \frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{3}{7} = \frac{18}{35}.
\end{aligned}$$

(c) Bayes' Theorem gives

$$P(B_1|R) = \frac{P(B_1) \cdot P(R|B_1)}{P(R)} = \frac{(1/2)(3/5)}{18/35} = \frac{7}{12} = 58.3\%.$$

[Remark: Before we see the chip, there is a 50% chance that it came from the first bowl. After we see that the chip is red, there is a 58.3% chance that it came from the first bowl.]

3. A diagnostic test is administered to a random person to determine if they have a certain disease. Consider the events

$$\begin{aligned} T &= \{\text{the test returns positive}\}, \\ D &= \{\text{the person has the disease}\}. \end{aligned}$$

Suppose that the test has the following "false positive" and "false negative" rates:

$$P(T|D') = 0.5\% \quad \text{and} \quad P(T'|D) = 0.2\%.$$

(a) For any events A, B recall that the Law of Total Probability says

$$P(A) = P(A \cap B) + P(A \cap B').$$

Use this to give an algebraic proof of the formula

$$1 = P(B|A) + P(B'|A).$$

(b) Use part (a) to compute the probability $P(T|D)$ of a "true positive" and the probability $P(T'|D')$ of a "true negative."

(c) Assume that 10% of the population has the disease, so that $P(D) = 10\%$. In this case compute the probability $P(T)$ that a random person tests positive. [Hint: The Law of Total Probability says $P(T) = P(T \cap D) + P(T \cap D')$.]

(d) Suppose that a random person is tested and the test returns positive. Compute the probability $P(D|T)$ that this person actually has the disease. Is this a good test?

(a) We divide both sides by $P(A)$ to get

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B') \\ \frac{P(A)}{P(A)} &= \frac{P(A \cap B)}{P(A)} + \frac{P(A \cap B')}{P(A)} \\ 1 &= P(B|A) + P(B'|A). \end{aligned}$$

(b) We have

$$P(T|D) = 1 - P(T'|D) = 99.8\% \qquad \text{and} \qquad P(T'|D') = 1 - P(T|D') = 99.5\%.$$

(c) The Law of Total Probability gives

$$\begin{aligned} T &= (T \cap D) \cup (T \cap D') \\ P(T) &= P(T \cap D) + P(T \cap D') \\ &= P(D) \cdot P(T|D) + P(D') \cdot P(T|D') \\ &= (0.1)(0.998) + (0.9)(0.005) \\ &= 10.43\%. \end{aligned}$$

(d) Bayes' Theorem gives

$$\begin{aligned}
P(D|T) &= \frac{P(D\cap T)}{P(T)} \\
&= \frac{P(D)\cdot P(T|D)}{P(D)\cdot P(T|D) + P(D')\cdot P(T|D')} \\
&= \frac{(0.1)(0.998)}{(0.1)(0.998)+(0.9)(0.005)} \\
&= 95.7\%.
\end{aligned}$$

Maybe this is a good test. I don't know.

4. Consider a coin with $P(H) = p$ and $P(T) = q$. Flip the coin until the first head shows up and then stop. Let X be the number of flip you made. The probability mass function and support of this *geometric random vabiable* are given by

$$P(X=k) = q^{k-1}p \qquad \text{and} \qquad S_X = \{1,2,3,\ldots\}.$$

(a) Use the geometric series $1+q+q^2+\cdots = (1-q)^{-1}$ to show that

$$\sum_{k\in S_X} P(X=k) = 1.$$

(b) Differentiate the geometric series to get $0+1+2q+3q^2+\cdots = (1-q)^{-2}$ and use this series to show that

$$E[X] = \sum_{k\in S_X} k\cdot P(X=k) = \frac{1}{p}.$$

(c) Application: Start rolling a fair 6-sided die. On average, how long do you have to wait until you see "1" for the first time?

Throughout I will assume that $0 < q = 1-p < 1$. (The cases $p\in\{0,1\}$ are slightly different.) For (a) we have

$$\begin{aligned}
\sum_{k\in S_X} P(X=k) &= \sum_{k=1}^{\infty} P(X=k) \\
&= \sum_{k=1}^{\infty} q^{k-1}p \\
&= p + qp + q^2p + q^3p + \cdots \\
&= p(1+q+q^2+q^3+\cdots) \\
&= p/(1-q) \\
&= p/p \\
&= 1.
\end{aligned}$$

In words: The total probability is 1. Then for (b) we have

$$\begin{aligned} E[X] = \sum_{k \in S_X} k \cdot P(X = k) &= \sum_{k=1}^{\infty} k \cdot P(X = k) \\ &= \sum_{k=1}^{\infty} k q^{k-1} p \\ &= p + 2qp + 3q^2 p + 4q^5 p + \cdots \\ &= p(1 + 2q + 3q^2 + 4q^3 + \cdots) \\ &= p/(1-q)^2 \\ &= p/p^2 \\ &= 1/p. \end{aligned}$$

In words: We expect to see the first H on the $(1/p)$-th flip of the coin.

(c) Application: For example, we can think of a fair six-sided die as a strange coin where $H = \{\text{we get 1}\}$ and $T = \{\text{we don't get 1}\}$, so that $P(H) = p = 1/6$. Let X be the number of rolls until we see "1." Then by part (b) we have

$$E[\text{number of rolls until our first 1}] = E[X] = \frac{1}{p} = \frac{1}{1/6} = 6.$$

That makes sense.

5. There are 2 red balls and 5 green balls in an urn. Suppose you grab 3 balls without replacement and let R be the number of red balls you get.

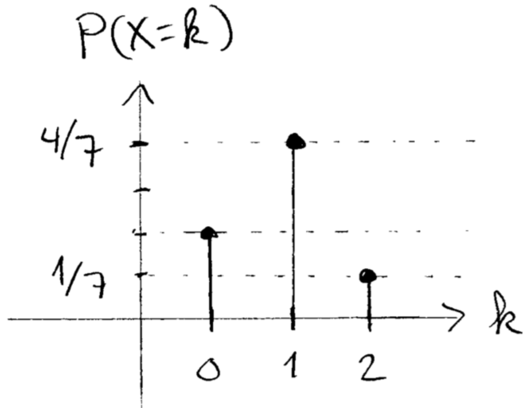
(a) What is the support of this random variable?
(b) Draw a picture of the probability mass function $f_R(k) = P(R = k)$.
(c) Compute the expected value $E[R]$. Does the answer make sense?

(a) The support is $S_X = \{0, 1, 2\}$.

(b) This X is a hypergeometric random variable with pmf given by the following table:

k	0	1	2
$P(X = k)$	$\frac{\binom{2}{0}\binom{5}{3}}{\binom{7}{3}} = \frac{2}{7}$	$\frac{\binom{2}{1}\binom{5}{2}}{\binom{7}{3}} = \frac{4}{7}$	$\frac{\binom{2}{2}\binom{5}{1}}{\binom{7}{3}} = \frac{1}{7}$

Here is the line graph:



(c) We have

$$E[X] = 0 \cdot P(X=0) + 1 \cdot P(X=2) + 2 \cdot P(X=2) = 0 \cdot \frac{2}{7} + 1 \cdot \frac{4}{7} + 2 \cdot \frac{1}{7} = \frac{5}{5} = \frac{6}{7}.$$

This answer makes sense because 2/7 of the balls in the urn are red. If we grab 3 balls, then we expect 2/7 of them, i.e., $3 \times 2/7 = 6/7$ balls, to be red. In general, suppose that there are r red and g green balls in an urn. If I grab n balls without replacement then I expect $n \times r/(r+g)$ of them to be red.

Also, the answer makes sense because it looks like a reasonable center of mass for the pmf.

6. Suppose that a "fair 3-sided die" has sides labeled $\{1, 2, 3\}$. Roll a pair of these dice and consider the following random variables:

$$X = \text{the number that shows up on the first roll,}$$
$$Y = \text{the number that shows up on the second roll.}$$

(a) Write down all elements of the sample space S.
(b) Compute the expected values $E[X]$ and $E[Y]$. [Hint: There is a shortcut.]
(c) Use the "linearity of expectation" to compute the expected values of the sum and difference: $E[X+Y]$ and $E[X-Y]$.
(d) Compute the expected value of the absolute value of the difference: $E[|X-Y|]$. [Hint: There is no shortcut. You must first compute the probability mass function $f_{|X-Y|}(k) = P(|X-Y| = k)$ for each value of k.]

(a) I will assume that the dice are ordered.[1] Here is the sample space:

$$S = \left\{ \begin{array}{ccc} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{array} \right\}$$

[1]Unordered dice makes the problem harder.

(b) Here are the probability mass functions for X and Y:

k	1	2	3
$P(X=k)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

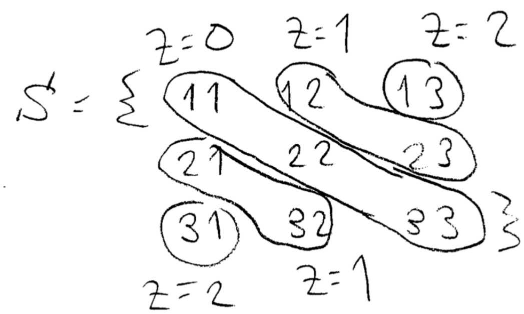
k	1	2	3
$P(Y=k)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

It follows that $E[X] = E[Y] = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 2$.

(c) Then from the "linearity of expectation" we obtain

$$E[X+Y] = E[X] + E[Y] = 2 + 2 = 4,$$
$$E[X-Y] = E[X] - E[Y] = 2 - 2 = 0.$$

(d) Now let $Z = |X - Y|$. There is no shortcut to compute $E[Z]$. First we circle the relevant outcomes:



Then we add them up:

k	0	1	2
$P(Z=k)$	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{2}{9}$

Then we compute the expected value:

$$E[Z] = 0 \cdot \frac{3}{9} + 1 \cdot \frac{4}{9} + 2 \cdot \frac{2}{9} = \frac{8}{9} = 0.89.$$

Interpretation: Roll a pair of fair 3-sided dice with sides labeled $\{1, 2, 3\}$. The difference of the two numbers that show up will be 0.89 (on average).