

1. An urn contains 3 red and 2 green balls. Your friend selects two balls (at random and without replacement) and tells you that at least one of the balls is red. What is the probability that the other ball is also red? [Hint: Let $A = \{1\text{st ball is red}\}$ and $B = \{2\text{nd ball is red}\}$. You want to compute the probability of $A \cap B$, assuming that $A \cup B$ happens.]

2. There are two bowls on a table. The first bowl contains 3 chips and 2 green chips. The second bowl contains 3 red chips and 4 green chips. Your friend walks up to the table and chooses one chip at random. Consider the events

$$B_1 = \{\text{chip comes from 1st bowl}\},$$

$$B_2 = \{\text{chip comes from 2nd bowl}\},$$

$$R = \{\text{chip is red}\}.$$

- Compute the forwards probabilities $P(R|B_1)$ and $P(R|B_2)$.
- Assuming that your friend is equally likely to choose either bowl (i.e., $P(B_1) = P(B_2) = 1/2$), compute the probability $P(R)$ that the chip is red.
- Compute the backwards probability $P(B_1|R)$. That is, assuming that your friend chose a red chip, what is the probability that it came from the first bowl?

3. A diagnostic test is administered to a random person to determine if they have a certain disease. Consider the events

$$T = \{\text{the test returns positive}\},$$

$$D = \{\text{the person has the disease}\}.$$

Suppose that the test has the following “false positive” and “false negative” rates:

$$P(T|D') = 0.5\% \quad \text{and} \quad P(T'|D) = 0.2\%.$$

- For any events A, B recall that the Law of Total Probability says

$$P(A) = P(A \cap B) + P(A \cap B').$$

Use this to give an algebraic proof of the formula

$$1 = P(B|A) + P(B'|A).$$

- Use part (a) to compute the probability $P(T|D)$ of a “true positive” and the probability $P(T'|D')$ of a “true negative.”
- Assume that 10% of the population has the disease, so that $P(D) = 10\%$. In this case compute the probability $P(T)$ that a random person tests positive. [Hint: The Law of Total Probability says $P(T) = P(T \cap D) + P(T \cap D')$.]
- Suppose that a random person is tested and the test returns positive. Compute the probability $P(D|T)$ that this person actually has the disease. Is this a good test?

4. Consider a coin with $P(H) = p$ and $P(T) = q$. Flip the coin until the first head shows up and then stop. Let X be the number of flip you made. The probability mass function and support of this *geometric random variable* are given by

$$P(X = k) = q^{k-1}p \quad \text{and} \quad S_X = \{1, 2, 3, \dots\}.$$

- (a) Use the geometric series $1 + q + q^2 + \cdots = (1 - q)^{-1}$ to show that

$$\sum_{k \in S_X} P(X = k) = 1.$$

- (b) Differentiate the geometric series to get $0 + 1 + 2q + 3q^2 + \cdots = (1 - q)^{-2}$ and use this series to show that

$$E[X] = \sum_{k \in S_X} k \cdot P(X = k) = \frac{1}{p}.$$

- (c) Application: Start rolling a fair 6-sided die. On average, how long do you have to wait until you see “1” for the first time?

5. There are 2 red balls and 5 green balls in an urn. Suppose you grab 3 balls without replacement and let R be the number of red balls you get.

- (a) What is the support of this random variable?
(b) Draw a picture of the probability mass function $f_R(k) = P(R = k)$.
(c) Compute the expected value $E[R]$. Does the answer make sense?

6. Suppose that a “fair 3-sided die” has sides labeled $\{1, 2, 3\}$. Roll a pair of these dice and consider the following random variables:

X = the number that shows up on the first roll,

Y = the number that shows up on the second roll.

- (a) Write down all elements of the sample space S .
(b) Compute the expected values $E[X]$ and $E[Y]$. [Hint: There is a shortcut.]
(c) Use the “linearity of expectation” to compute the expected values of the sum and difference: $E[X + Y]$ and $E[X - Y]$.
(d) Compute the expected value of the absolute value of the difference: $E[|X - Y|]$. [Hint: There is no shortcut. You must first compute the probability mass function $f_{|X - Y|}(k) = P(|X - Y| = k)$ for each value of k .]