Math 224 **Spring 2022**
Homework 6 **Drew Armstrong**

---

1. Rounding Error. Your friend has a list of ten real numbers, whose values are unknown to you: $x_1, \ldots, x_{10} \in \mathbb{R}$. Your friend rounds each number to the nearest integer and sends you the results: $X_1, X_2, \ldots, X_{10} \in \mathbb{Z}$. We will assume that $X_i = x_i + U_i$, where each U_i is a uniform random variable on the interval $[-1/2, 1/2]$.

(a) Compute $E[U_i]$ and $\text{Var}(U_i)$.
(b) Consider the sum of the rounded numbers $X = X_1 + \cdots + X_{10}$ and the sum of the unrounded numbers $x = x_1 + \cdots + x_{10}$. Prove that $E[X] = x$.
(c) Assuming that the random variables U_i are independent, use the CLT to estimate the probability that $|X - x| > 1/2$. [Hint: The CLT says that $X - x = U_1 + \cdots + U_{10}$ is approximately normal. You just need to compute $E[X - x]$ and $\text{Var}(X - x)$.]

2. Tail Probabilities. Consider a standard normal variable $Z \sim N(0, 1)$. Solve for a.

(a) $P(Z > a) = 93\%$
(b) $P(Z < a) = 35\%$
(c) $P(|Z| > a) = 2\%$
(d) $P(|Z| < a) = 80\%$

3. A Bernoulli Hypothesis Test. A six-sided die has sides labeled $\{1, 2, 3, 4, 5, 6\}$. Let p be the probability of getting a 6. Before performing any experiments we will assume that $H_0 =$ "$p = 1/6$" is true. Now suppose that you roll the die 600 times and let Y be the number of times you get 6. Which values of Y would cause you to reject H_0 in favor of $H_1 =$ "$p > 1/6$" at the 99% level of confidence? That is, what is the rejection region?

4. Confidence Intervals for a Proportion. Let p be the proportion of Americans who are left-handed. In order to estimate p, we randomly selected $n = 1000$ Americans and we found that $Y = 125$ of them are left-handed. Use this information to compute two-sided, symmetric $(1 - \alpha)100\%$ confidence intervals for p when $\alpha = 5\%$, 2.5% and 1%.

5. Sample Variance. Consider an iid sample $X_1, \ldots, X_n$ with unknown mean μ and unknown variance σ^2. In order to estimate σ^2 we define the *sample variance* as follows:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2,$$

where $\overline{X} = (\sum_{i=1}^n X_i)/n$ is the usual sample mean.

(a) Show that $\sum_{i=1}^n \left(X_i - \overline{X}\right)^2 = \left(\sum_{i=1}^n X_i^2\right) - n\overline{X}^2$. [Hint: $n\overline{X} = \sum_{i=1}^n X_i$.]
(b) Show that $E[X_i^2] = \mu^2 + \sigma^2$ and $E[\overline{X}^2] = \mu^2 + \sigma^2/n$. [Hint: By definition we have $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$, which implies that $E[\overline{X}] = \mu$ and $\text{Var}(\overline{X}) = \sigma^2/n$.]
(c) Combine (a) and (b) to show that $E[S^2] = \sigma^2$. This is why the definition of S^2 has $n - 1$ in the denominator instead of n.

6. A Small Sample. The label weight of a Cadbury Creme Egg is 1.2oz. In order to test this you weighed 10 eggs and obtained the following values (in ounces):

1.12	1.01	1.04	1.10	1.00	1.04	1.28	1.17	1.19	1.24

Let X represent the underlying distribution with unknown mean $\mu = E[X]$. For simplicity we assume that X is normal.

(a) Compute the sample mean $\overline{X}$ and the sample variance S^2.

(b) Look up the t-tail probabilities $t_{5\%}(9)$ and $t_{2.5\%}(9)$.

(c) Test the hypothesis $H_0 =$ "$\mu = 1.2$" against the one-sided alternative $H_1 =$ "$\mu < 1.2$" at the 5% level of significance.

(d) Compute a two-sided symmetric 95% confidence interval for the unknown μ.