Math 224 **Spring 2022**
Homework 5 **Drew Armstrong**

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1. Uniform Random Variable. Let U be the uniform random variable on the interval $[2,6]$. Compute the following:

$$P(3 < U < 4), \quad P(3 < U < 7), \quad \mu = E[U], \quad \sigma^2 = \text{Var}(U), \quad P(\mu - \sigma < U < \mu + \sigma).$$

Recall that the density of U is

$$f_U(x) = \begin{cases} 1/4 & 2 \leq x \leq 6, \\ 0 & \text{otherwise.} \end{cases}$$

The idea is that the density is **constant** on the interval $[2,6]$. We choose $1/4 = 1/(6-2)$ so that the total area is 1. Here is a picture:

To compute each probability we must integrate the density. First we have

$$P(3 < U < 4) = \int_3^4 f_U(x)\,dx = \int_3^4 (1/4)\,dx = 1/4.$$

We must break up the second interval because the density has a piecewise definition:

$$P(3 < U < 7) = \int_3^7 f_U(x)\,dx = \int_3^6 1/4\,dx + \int_6^7 0\,dx = 3/4 + 0 = 3/4.$$

Remark: The integral of a constant is $\int_a^b c\,dx = c[x]_a^b = c(b-a)$. We can also view this as the area of a rectangle with base $b-a$ and height c.

The first moment is

$$\begin{aligned} E[U] &= \int_{-\infty}^{\infty} x \cdot f_U(x)\,dx \\ &= \int_{-\infty}^{2} 0\,dx + \int_2^6 x \cdot (1/4)\,dx + \int_6^{\infty} 0\,dx. \\ &= (1/4)[x^2/2]_2^6 \\ &= (1/4)[36/2 - 4/2] \\ &= 4, \end{aligned}$$

which we could have guessed because the distribution is symmetric about 4. The second moment is

$$
\begin{aligned}
E[U^2] &= \int_{-\infty}^{\infty} x^2 \cdot f_U(x)\,dx \\
&= \int_2^6 x^2(1/4)\,dx \\
&= (1/4)[x^3/3]_2^6 \\
&= (1/4)[216/3 - 8/3] \\
&= 52/3,
\end{aligned}
$$

and hence the variance is

$$\text{Var}(U) = E[U^2] - E[U]^2 = 52/3 - (4)^2 = 4/3.$$

Finally, we compute the probability that U falls between $\mu - \sigma$ and $\mu + \sigma$. Since $\mu = 4$ and $\sigma = \sqrt{4/3}$ we have

$$
\begin{aligned}
P(\mu - \sigma < U < \mu + \sigma) &= P(4 - \sqrt{4/3} < U < 4 + \sqrt{4/3}) \\
&= \int_{4-\sqrt{4/3}}^{4+\sqrt{4/3}} (1/4)\,dx \\
&= (1/4)[x]_{4-\sqrt{4/3}}^{4+\sqrt{4/3}} \\
&= (1/4)[(7/2 + \sqrt{3/4}) - (7/2 - \sqrt{3/4})] \\
&= (1/4)[2\sqrt{4/3}] \\
&\approx 57.7\%.
\end{aligned}
$$

Remark: We would have gotten 57.7% for the uniform random variable on **any** interval $[a, b]$. See the notes for a proof.

2. A Continuous Random Variable. Let X be a continuous random variable with the following density:

$$f_X(x) = \begin{cases} c(1 - x^4) & -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the correct value of the constant c.
(b) Compute $\mu = E[X]$ and $\sigma^2 = \text{Var}(X)$.
(c) Compute $P(\mu - \sigma < X < \mu + \sigma)$.
(d) Draw a picture of the whole situation.

(a): The total mass of a probability density must be 1:

$$1 = \int_{-1}^{1} c \cdot (1 - x^4)\,dx = c[x - x^5/5]_{-1}^{1} = c[(1 - 1/5) - (-1 + 1/5)] = c[8/5].$$

Hence we must have $c = 5/8$.

(b): The mean is

$$\begin{aligned}\mu = E[X] &= \int_{-1}^{1} x \cdot (5/8)(1-x^4)\,dx \\ &= \int_{-1}^{1} (5/8)(x-x^5)\,dx \\ &= (5/8)[x^2/2 - x^6/6]_{-1}^{1} \\ &= (5/8)[(1/2-1/6)-(1/2-1/6)] \\ &= (5/8)[0] \\ &= 0,\end{aligned}$$

which we could have predicted because the distribution is symmetric about 0. The second moment is

$$\begin{aligned}E[X^2] &= \int_{-1}^{1} x^2 \cdot (5/8)(1-x^4)\,dx \\ &= \int_{-1}^{1} (5/8)(x^2-x^6)\,dx \\ &= (5/8)[x^3/3 - x^7/7]_{-1}^{1} \\ &= (5/8)[(1/3-1/7)-(-1/3+1/7)] \\ &= (5/8)[5/12] \\ &= 5/21,\end{aligned}$$

and hence the variance is

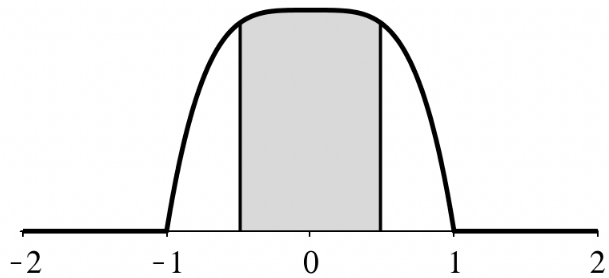
$$\sigma^2 = \text{Var}(X) = E[X^2] - E[X]^2 = 5/21 - 0^2 = 5/21.$$

(c): Finally, we compute the probability that X falls between $\mu - \sigma$ and $\mu + \sigma$. We will use $\mu = 0$ but we will leave $\sigma = \sqrt{5/12}$ unevaluated until the end of the calculation:

$$\begin{aligned}P(\mu-\sigma < X < \mu+\sigma) &= P(-\sigma < X < \sigma) \\ &= \int_{-\sigma}^{\sigma} (5/8)(1-x^4)\,dx \\ &= (5/8)[x - x^5/5]_{-\sigma}^{\sigma} \\ &= (5/8)[(\sigma - \sigma^5/5) - (-\sigma + \sigma^5/5)] \\ &= (5/8)[2\sigma - 2\sigma^5/5] \\ &= 2(5/8)\sigma[1-\sigma^4/5] \\ &= (10/8)\sqrt{5/21}[1-(5/12)^2/5]] \\ &= (545/441)\sqrt{5/21} \\ &= 60.3\%.\end{aligned}$$

Note that 60.3% is greater than 57.7% because this random variable is more concentrated near its mean than the uniform distribution in Problem 1.

(d): Here is a picture:



3. The Exponential Distribution. Fix some positive real number $\lambda > 0$ and let X be a continuous random variable with *exponential density*:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

(a) Verify that $\int f_X(x)\, dx = 1$. [Hint: Note that $e^{-\lambda x} \to 0$ as $x \to +\infty$.]
(b) Use integration by parts to compute $E[X]$.

(a): For any constant a we have $\int e^{ax}\, dx = e^{ax}/a$. Hence

$$\begin{aligned} \int_0^\infty \lambda e^{-\lambda x}\, dx &= \lambda[e^{-\lambda x}/(-\lambda)]_0^\infty \\ &= \lambda[0 + e^{-\lambda 0}/\lambda] \\ &= \lambda[0 + 1/\lambda] \\ &= 1. \end{aligned}$$

(b): The expected value is defined by

$$E[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx.$$

In order to compute this we use integration by parts. For any functions u and v, the product rule for differentials says that

$$d(uv) = udv + vdu.$$

Then integrating both sides gives the formula for integration by parts:

$$uv = \int udv + \int vdu,$$

or equivalently

$$\int udv = uv + \int vdu.$$

In our case we will take $u = x$ and $dv = \lambda e^{-\lambda x} dx$ so that $du = dx$ and $v = -e^{-\lambda x}$. Then[1]

$$\begin{aligned} E[X] &= \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx \\ &= \int_0^\infty u dv \\ &= [uv]_0^\infty - \int_0^\infty v du \\ &= [-x \cdot e^{-\lambda x}]_0^\infty - \int_0^\infty -e^{-\lambda x}\, dx \\ &= [0-0] - [-e^{-\lambda}/(-\lambda)]_0^\infty \\ &= [0-0] - [0 - 1/\lambda] \\ &= 1/\lambda. \end{aligned}$$

Remark: We can view the (continuous) exponential random variable as a limit of (discrete) geometric random variables. Consider a coin with $P(H) = p$ and let G be the number of coin flips until we see heads for the first time. If $P(H) = p$ then the expected number of coin flips is $E[G] = 1/p$. In this problem we can think of X as the amount of time we have to wait until a certain radioactive particle decays. This decay is controlled by a *decay constant* $\lambda > 0$ and the expected waiting time until decay is $E[X] = 1/\lambda$. Without going into details, we can think of a radioactive particle as a "continuously flipping coin", where heads means "decay" and tails means "don't decay", so that X is the amount of time we have to wait until we see heads. This interpretation emphasizes a strange quantum property of radioactive particles: they are *memoryless*. If a particle has not decayed for 100 years, this does not increase the chance that it will decay tomorrow.

4. Table of Z-Scores. Let $Z \sim N(0,1)$ so that $P(Z \le z) = \Phi(z)$. Use the attached table to compute the following probabilities:

(a) $P(Z < -0.3)$
(b) $P(0.25 < Z < 1.25)$
(c) $P(Z > 1)$, $P(Z > 2)$, $P(Z > 3)$
(d) $P(|Z| < 1)$, $P(|Z| < 2)$, $P(|Z| < 3)$

We repeatedly use the following facts:

$$P(z_1 < Z < z_2) = \Phi(z_2) - \Phi(z_1), \quad \Phi(z < Z) = 1 - \Phi(z), \quad P(-z) = 1 - \Phi(z).$$

(a): $P(Z < -0.3) = 1 - \Phi(0.3) = 1 - (0.6179) = 30.21\%$

(b): $P(0.25 < Z < 1.25) = \Phi(1.25) - \Phi(0.25) = (0.8944) - (0.5987) = \%29.57$

(c): For each we use the formula $P(Z > z) = 1 - \Phi(z)$:

$$\begin{aligned} \Phi(Z > 1) &= 1 - \Phi(1) = 1 - (0.8413) = 15.85\% \\ \Phi(Z > 2) &= 1 - \Phi(2) = 1 - (0.9772) = 2.28\% \\ \Phi(Z > 3) &= 1 - \Phi(3) = 1 - (0.9987) = 0.13\% \end{aligned}$$

[1] Here we use the fact that $x \cdot e^{-\lambda x} \to 0$ as $x \to \infty$. I should have included this as a hint.

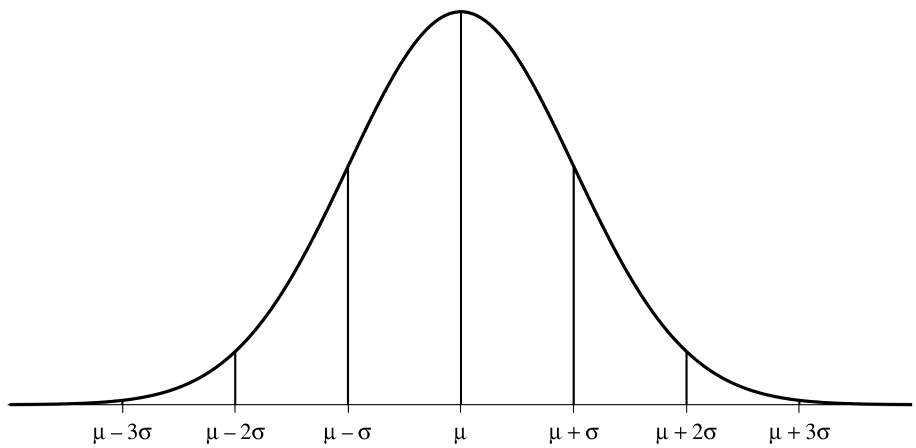
(d): First we note that

$$P(|Z| < z) = P(-z < Z < z) = \Phi(z) - \Phi(-z) = \Phi(z) - [1 - \Phi(z)] = 2\Phi(z) - 1.$$

Thus we have:

$$\begin{aligned}
\Phi(|Z| < 1) &= 2\Phi(1) - 1 = 2(0.8413) - 1 = 68.26\% \\
\Phi(|Z| < 2) &= 2\Phi(2) - 1 = 2(0.9772) - 1 = 95.44\% \\
\Phi(|Z| < 3) &= 2\Phi(3) - 1 = 2(0.9987) - 1 = 99.74\%
\end{aligned}$$

In summary, the probability that a normal random variable falls within 1, 2 or 3 standard deviations of its mean is approximately 68%, 95% and 99.7%, respectively. Here is a picture:



Remark: Here are some useful general formulas for constants a, b with $b > 0$:

$$\begin{aligned}
P(|Z - a| < b) &= P(a - b < Z < a + b), \\
P(|Z - a| > b) &= P(Z < a - b) + P(Z > a + b).
\end{aligned}$$

5. The de Moivre-Laplace Theorem. Consider a coin with $p = P(H) = 35\%$. Suppose that you flip the coin 100 times and let X be the number of times you get heads.

(a) Compute $E[X]$ and $\text{Var}(X)$.
(b) The de Moivre-Laplace Theorem says that X is approximately normal. Use this to estimate the probability $P(34 \leq X \leq 36)$. Don't forget to use a continuity correction.

(a): Since X has a binomial distribution with parameters $n = 100$ and $p = 0.35$ we know that

$$E[X] = np = 35 \quad \text{and} \quad \text{Var}(X) = npq = 22.75.$$

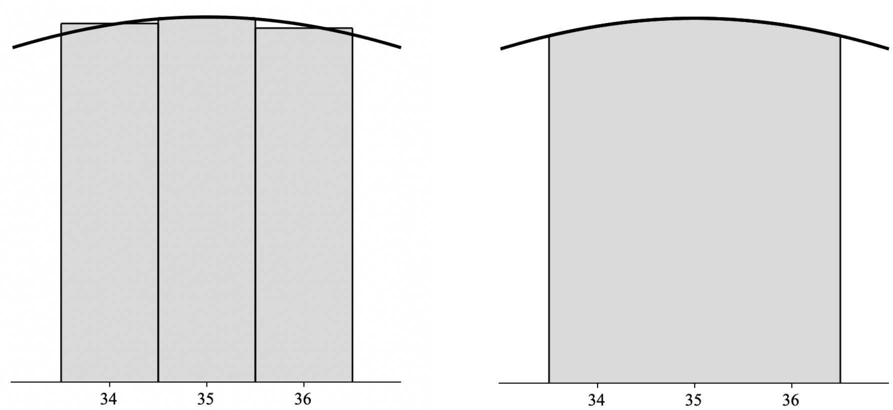
(b): Since np and nq are both reasonably large,[2] the de Moivre-Laplace theorem tells us that X is approximately normal. Let X' be a normal random variable with the same mean and

[2] The rule of thumb says that np and nq should both be greater than 10. Basically, we need n to be large and we need p not too close to 0 or 1.

variance: $X' \sim N(35, 22.75)$. Then we have

$$\begin{aligned}
P(34 \leq X \leq 36) &\approx P(33.5 < X' < 36.5) \\
&= P\left(\frac{33.5-35}{\sqrt{22.75}} < \frac{X'-35}{\sqrt{22.75}} < \frac{36.5-35}{\sqrt{22.75}}\right) \\
&\approx P(-0.31 < Z < 0.31) \\
&= \Phi(0.31) - \Phi(-0.31) \\
&= \Phi(0.31) - [1 - \Phi(0.31)] \\
&= 2\Phi(0.31) - 1 \\
&= 2(0.6217) - 1 \\
&= 24.34\%.
\end{aligned}$$

In the first step we used a continuity correction, as described by the following picture:



My computer says that the exact probability is 24.66%, so our approximation is pretty good.[3]

6. The Central Limit Theorem. Let $X_1, X_2, \ldots, X_{180}$ be a sequence of iid[4] random variables with mean $\mu = 17$ and variance $\sigma^2 = 5$. Consider the sample mean

$$\overline{X} = \frac{1}{180}(X_1 + X_2 + \cdots + X_{180}).$$

(a) Compute $E[\overline{X}]$ and $\mathrm{Var}(\overline{X})$.
(b) The Central Limit Theorem tells us that $\overline{X}$ is approximately normal. Use this fact together with part (a) to estimate the probability $P(\overline{X} > 17.3)$.

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[3]If we hadn't used the continuity correction above then we would have found 32.56%, which is pretty bad.

[4]Independent and identically distributed. This means that the X_i are jointly independent and each has the same density function (which is unknown to us).

(a): We use linearity to compute the expected value:

$$\begin{aligned}
E[\overline{X}] &= E\left[\frac{1}{180}(X_1 + X_2 + \cdots + X_{180})\right] \\
&= \frac{1}{180}\left(E[X_1] + E[X_2] + \cdots + E[X_{180}]\right) \\
&= \frac{1}{180}(17 + 17 + \cdots + 17) \\
&= 17.
\end{aligned}$$

To compute the variance, we use the fact that the X_i are independent so that the variance of the sum is the sum of the variances. We also use the fact that $\text{Var}(aX) = a^2\text{Var}(X)$ for all constants a and random variables X:

$$\begin{aligned}
\text{Var}(\overline{X}) &= \text{Var}\left(\frac{1}{180}(X_1 + X_2 + \cdots + X_{180})\right) \\
&= \left(\frac{1}{180}\right)^2 \left(\text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_{180})\right) \\
&= \left(\frac{1}{180}\right)^2 (5 + 5 + \cdots + 5) \\
&= 5/180 \\
&= 1/36.
\end{aligned}$$

(b): The Central Limit Theorem tells us that $\overline{X}$ is approximately $N(17, 1/36)$, so that $Z := (\overline{X} - 17)/\sqrt{1/36}$ is approximately $N(0, 1)$. Thus we have

$$\begin{aligned}
P(\overline{X} > 17.3) &= P\left(\frac{\overline{X} - 17}{\sqrt{1/36}} > \frac{17.3 - 17}{\sqrt{1/36}}\right) \\
&= P(Z > 1.8) \\
&= 1 - P(Z < 1.8) \\
&= 1 - \Phi(1.8) \\
&= 1 - (0.9641) \\
&= 3.59\%.
\end{aligned}$$