Math 224 **Spring 2022**
Homework 5 **Drew Armstrong**

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1. Uniform Random Variable. Let U be the uniform random variable on the interval $[2,6]$. Compute the following:

$$P(3 < U < 4), \quad P(3 < U < 7), \quad \mu = E[U], \quad \sigma^2 = \text{Var}(U), \quad P(\mu - \sigma < U < \mu + \sigma).$$

2. A Continuous Random Variable. Let X be a continuous random variable with the following density:

$$f_X(x) = \begin{cases} c(1-x^4) & -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the correct value of the constant c.
(b) Compute $\mu = E[X]$ and $\sigma^2 = \text{Var}(X)$.
(c) Compute $P(\mu - \sigma < X < \mu + \sigma)$.
(d) Draw a picture of the whole situation.

3. The Exponential Distribution. Fix some positive real number $\lambda > 0$ and let X be a continuous random variable with *exponential density*:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

(a) Verify that $\int f_X(x)\,dx = 1$. [Hint: Note that $e^{-\lambda x} \to 0$ as $x \to +\infty$.]
(b) Use integration by parts to compute $E[X]$.

4. Table of Z-Scores. Let $Z \sim N(0,1)$ so that $P(Z \leq z) = \Phi(z)$. Use the attached table to compute the following probabilities:

(a) $P(Z < -0.3)$
(b) $P(0.25 < Z < 1.25)$
(c) $P(Z > 1)$, $P(Z > 2)$, $P(Z > 3)$
(d) $P(|Z| < 1)$, $P(|Z| < 2)$, $P(|Z| < 3)$

5. The de Moivre-Laplace Theorem. Consider a coin with $p = P(H) = 35\%$. Suppose that you flip the coin 100 times and let X be the number of times you get heads.

(a) Compute $E[X]$ and $\text{Var}(X)$.
(b) The de Moivre-Laplace Theorem says that X is approximately normal. Use this to estimate the probability $P(34 \leq X \leq 36)$. Don't forget to use a continuity correction.

6. The Central Limit Theorem. Let $X_1, X_2, \ldots, X_{180}$ be a sequence of iid[1] random variables with mean $\mu = 17$ and variance $\sigma^2 = 5$. Consider the sample mean

$$\overline{X} = \frac{1}{180}(X_1 + X_2 + \cdots + X_{180}).$$

(a) Compute $E[\overline{X}]$ and $\text{Var}(\overline{X})$.
(b) The Central Limit Theorem tells us that $\overline{X}$ is approximately normal. Use this fact together with parts (a) and (b) to estimate the probability $P(\overline{X} > 17.3)$.

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[1] Independent and identically distributed. This means that the X_i are jointly independent and each has the same density function (which is unknown to us).