Math 224 **Spring 2022**
Homework 4 **Drew Armstrong**

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1. Bilinearity of Covariance. Let X and Y be random variables on the same experiment with the following moments:

$$E[X]=1, \quad E[X^2]=2, \quad E[Y]=2, \quad E[Y^2]=6, \quad E[XY]=5.$$

(a) Compute $\text{Var}(X)$, $\text{Var}(Y)$ and $\text{Cov}(X,Y)$.
(b) Use part (a) to compute $\text{Cov}(2X-Y, 3X+7Y)$.

2. Standardization. Let X be a random variable with $E[X]=\mu$ and $\text{Var}(X)=\sigma^2$. Consider the random variable[1]

$$X' = \frac{X-\mu}{\sigma}.$$

(a) Use linearity of expectation to compute $E[X']$
(b) Use properties of variance to compute $\text{Var}(X')$.

3. Joint Distributions. Let X and Y be random variables on the same experiment. Suppose that X and Y have the following joint pmf table:[2]

$X \setminus Y$	0	1	3	
-1	1/12	1/12	2/12	4/12
1	2/12	3/12	3/12	8/12
	3/12	4/12	5/12	

(a) Compute $E[X]$ and $E[Y]$.
(b) Compute $\text{Var}(X)$ and $\text{Var}(Y)$.
(c) Compute $E[XY]$ and $\text{Cov}(X,Y)$.

4. Multinomial Covariance. Consider a fair 3-sided die with sides labeled $\{a,b,c\}$. Roll the die 3 times and consider the following random variables:

$$A = \text{the number of times that } a \text{ shows up,}$$
$$B = \text{the number of times that } b \text{ shows up.}$$

(a) Write out the joint pmf table of A and B. [Hint: Recall the formula

$$P(A=k, B=\ell) = \frac{3!}{k!\ell!(3-k-\ell)!}(1/3)^k(1/3)^\ell(1/3)^{3-k-\ell}.]$$

(b) Use the joint pmf table to compute $\text{Cov}(A,B)$. Observe that it is negative. Indeed, if the number of a's goes up then the number of b's has a tendency to go down (and vice versa) because the total number of rolls is fixed.

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[1] This is not a derivative. I just didn't want to waste another letter of the alphabet.
[2] For example, the table says that $P(X=-1, Y=3) = 2/12$ and $P(Y=3) = 5/12$.

5. The Hat Check Problem. Suppose that n people go to a party and leave their hats with the hat check person.[3] At the end of the party the hat check person returns the hats randomly. Consider the following Bernoulli variables:

$$X_i = \begin{cases} 1 & \text{if the } i\text{th person gets their own hat back,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = X_1 + \cdots + X_n$ be the total number of people who get their own hat back.

(a) Compute $E[X_i]$ and $\text{Var}(X_i)$ for any i. [Hint: Compute $P(X_i = 1)$.]

(b) Use linearity to compute the expected value $E[X]$.

(c) Compute the mixed moment $E[X_iX_j]$ for $i \neq j$. [Hint: Note that

$$X_iX_j = \begin{cases} 1 & \text{if the } i\text{th and } j\text{th persons both get their own hat back,} \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $P(X_iX_j = 1) = P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1|X_i = 1)$.]

(d) Use parts (a) and (c) to compute the covariance $\text{Cov}(X_i, X_j) = E[X_iX_i] - E[X_i]E[X_j]$ and the variance $\text{Var}(X)$. [Hint: Bilinearity and symmetry of covariance gives

$$\begin{aligned} \text{Var}(X) &= \text{Cov}(X_1 + \cdots + X_n, X_1 + \cdots + X_n) \\ &= \sum_{i,j} \text{Cov}(X_i, X_j) \\ &= \sum_{i} \text{Cov}(X_i, X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j) \\ &= \sum_{i} \text{Cov}(X_i, X_i) + 2\sum_{i<j} \text{Cov}(X_i, X_j) \\ &= \sum_{i} \text{Var}(X_i) + 2\sum_{i<j} \text{Cov}(X_i, X_j). \end{aligned}$$

The number of pairs in the second sum is $\binom{n}{2} = n(n-1)/2$.]

[3] Long ago people used to wear hats, but not indoors.