Math 224 **Spring 2022**
Homework 3 **Drew Armstrong**

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1. Functions of a Random Variable. Let X be the number of heads obtained in 3 flips of a fair coin. Compute the following expected values:

(a) $E[X^2]$,
(b) $E[2^X]$,
(c) $E[e^{tX}]$, where t is some constant.

[Hint: The general formula is $E[g(X)] = \sum_k g(k) \cdot P(X = k)$.]

2. Tricks With the Binomial Theorem. Let X be a binomial random variable with parameters (n, p) and let z be any real number. The following function $G(z)$ is called the *probability generating function of* X:

$$G(z) = \sum_{k=0}^{n} P(X = k) \cdot z^k.$$

(a) Prove that $G'(1) = E[X]$. [This is the derivative of $G(z)$ evaluated at $z = 1$.]
(b) Use the Binomial Theorem to prove that $G(z) = (pz + q)^n$.
(c) Combine parts (a) and (b) to prove that $E[X] = np$.

3. Sums of Random Variables. A fair coin has sides labeled $\{1, 3\}$. Suppose that you flip the coin three times and consider the random variables X_1, X_2, X_3 defined by

$$X_i = \text{the number that shows up on the } i\text{th coin flip.}$$

(a) Compute $E[X_1]$, $E[X_2]$ and $E[X_3]$.
(b) Let $X = X_1 + X_2 + X_3$ be the sum of the three numbers you get. Compute the probability mass function $P(X = k)$ and draw the probability histogram of X. [Hint: There is no shortcut. You must write down all 8 elements of the sample space.]
(c) Compute $E[X]$ using the pmf for X from part (b).
(d) Compute $E[X]$ using linearity and your answer from part (a).

4. Roulette. A European roulette wheel has 37 pockets: 1 colored green, 18 colored red and 18 colored black. To play the game you pay \$1 and pick a color from $\{r, b, g\}$ (red, black, green). Then the croupier[1] spins the wheel and observes which pocket a marble falls into. Your winnings are decided by the following rules:

- If you pick r and the marble lands in a red pocket you win \$2.
- If you pick b and the marble lands in a black pocket you win \$2.
- If you pick g and the marble lands in the green pocket you win \$35.
- Otherwise you get nothing.

Suppose that the 37 pockets are equally likely and let W be your winnings.

(a) Compute $E[W]$ if you pick r. [You get the same answer for color b.]
(b) Compute $E[W]$ if you pick g.
(c) Compute $E[W]$ if you pick from $\{r, b, g\}$ at random, each with probability $1/3$.

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[1] The person who spins the wheel.

[Hint: In each case the random variable W has support $\{0, 2, 35\}$, hence you need to compute the probabilities $P(W = 0)$, $P(W = 2)$ and $P(W = 35)$.]

5. Geometric Random Variables. Let X be a geometric random variable with parameters p and $q = 1 - p$, so that $P(X = k) = pq^{k-1}$. We can interpret X as the number of coin flips until we see H for the first time, where $P(H) = p$. Let's assume that $p \neq 0$ so that $q < 1$.

(a) Use the geometric series to verify that $\sum_k P(X = k) = 1$.
(b) If $k \geq 0$, use the geometric series to prove that $P(X \geq k) = q^{k-1}$.
(c) Differentiate the geometric series to prove that $E[X] = 1/p$.

6. The Coupon Collector Problem. Suppose that you roll a fair n-sided die until you see all n sides, and let X be the number of rolls that you did. In this problem I will guide you through a method to compute the expected number of rolls $E[X]$.

(a) Fix some $0 \leq k \leq n-1$ and suppose that you have already seen k sides of the die. Let X_k be the number of die rolls until you see one of the remaining $n - k$ sides. Compute $E[X_k]$. [Hint: This is a geometric random variable. Think of the die as a coin with $T =$"you see one of the k sides that you've already seen" and $H =$"you see one of the $n - k$ sides that you haven't seen yet". Use Problem 5(c).]
(b) We observe that $X = X_0 + X_1 + \cdots + X_{n-1}$ is the total number of rolls until you see all n sides of the die. Use part (a) and linearity of expectation to compute $E[X]$.
(c) Example: Suppose that you roll a fair 6-sided die until you see all six sides. On average, how many rolls do you expect to make?