Math 224 **Exam 2**
Spring 2022 **Thurs Mar 24**

This is a closed book test. No electronic devices are allowed. There are 5 pages and 5 problems, each worth 6 points, for a total of 30 points.

Problem 1. Let X be a random variable with the following pmf table:

k	-1	0	1	2
$P(X=k)$	$3/8$	$2/8$	$2/8$	$1/8$

(a) Compute $E[X]$.

$$E[X] = (-1)\frac{3}{8} + (0)\frac{2}{8} + (1)\frac{2}{8} + (2)\frac{1}{8} = \frac{1}{8}$$

(b) Compute $E[X^2]$ and $\mathrm{Var}(X)$.

$$E[X^2] = (-1)^2\frac{3}{8} + (0)^2\frac{2}{8} + (1)^2\frac{2}{8} + (2)^2\frac{1}{8} = \frac{9}{8}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = 9/8 - (1/8)^2 = \frac{72}{64} - \frac{1}{64} = \frac{71}{64}$$

(c) Use the formula for $E[g(X)]$ to compute $E[2^X]$.

$$\begin{aligned} E[2^X] &= 2^{-1}\cdot\frac{3}{8} + 2^0\cdot\frac{2}{8} + 2^1\cdot\frac{2}{8} + 2^2\cdot\frac{1}{8} \\ &= \frac{3}{16} + \frac{2}{8} + \frac{4}{8} + \frac{4}{8} \\ &= \frac{3}{16} + \frac{4}{8} + \frac{8}{16} + \frac{8}{16} = \frac{23}{16} \end{aligned}$$

Problem 2. Let X and Y be random variables with the following joint pmf table:

$X \setminus Y$	0	1	2	
-1	$2/8$	$1/8$	$2/8$	$5/8$
1	$1/8$	$2/8$	0	$3/8$
	$3/8$	$3/8$	$2/8$	

(a) Compute $E[X]$ and $E[Y]$.

$$E[X] = (-1)\frac{5}{8} + (1)\frac{3}{8} = -\frac{2}{8}$$

$$E[Y] = (0)\frac{3}{8} + (1)\frac{3}{8} + (2)\frac{2}{8} = \frac{7}{8}$$

(b) Compute $E[XY]$ and $\text{Cov}(X, Y)$.

$$\begin{aligned} E[XY] &= (-1)(0)\frac{2}{8} + (-1)(1)\frac{1}{8} + (-1)(2)\frac{2}{8} \\ &\quad + (1)(0)\frac{1}{8} + (1)(1)\frac{2}{8} + (1)(2) \cdot 0 \\ &= -\frac{1}{8} - \frac{4}{8} + \frac{2}{8} = -\frac{3}{8} \end{aligned}$$

$$\text{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y] = -\frac{3}{8} - \left(-\frac{2}{8}\right)\left(\frac{7}{8}\right) = -\frac{24}{64} + \frac{14}{64} = -\frac{10}{64}$$

(c) Compute the probability $P(X + Y = 1)$.

$$P(X + Y = 1) = P(X = -1, Y = 2) + P(X = 1, Y = 0) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$$

Problem 3. Let X, Y be random variables with the following moments:

$$E[X] = 1, \quad E[X^2] = 3, \quad E[Y] = 2, \quad E[Y^2] = 5, \quad E[XY] = 1.$$

(a) Compute $\text{Cov}(X, Y)$.

$$\text{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y] = 1 - 1 \cdot 2 = -1$$

(b) Compute $E[(X + Y)^2]$. [Hint: Use linearity of expectation.]

$$\begin{aligned} E[(X + Y)^2] &= E[X^2 + 2XY + Y^2] \\ &= E[X^2] + 2 \cdot E[XY] + E[Y^2] \\ &= 2 + 2 \cdot 1 + 5 \\ &= 9 \end{aligned}$$

(c) Compute $\text{Var}(X - Y)$. [Hint: Use bilinearity of covariance.]

$$\begin{aligned} \text{Var}(X - Y) &= \text{Cov}(X - Y, X - Y) \\ &= \text{Cov}(X, X) - 2 \cdot \text{Cov}(X, Y) + \text{Cov}(Y, Y) \\ &= \text{Var}(X) + \text{Var}(Y) - 2 \cdot \text{Cov}(X, Y) \\ &= \big(E[X^2] - E[X]^2\big) + \big(E[Y^2] - E[Y]^2\big) - 2 \cdot \text{Cov}(X, Y) \\ &= (2 - 1^2) + (5 - 2^2) - 2 \cdot (-1) \\ &= 1 + 1 + 2 \\ &= 4 \end{aligned}$$

Remark: I changed this problem in the solutions to make it more meaningful to future readers. On the exam I gave

$$E[X] = 1, \quad E[X^2] = 2, \quad E[Y] = 2, \quad E[Y^2] = 3, \quad E[XY] = 3,$$

which is no good because it makes the variance of Y negative: $\text{Var}(Y) = 3 - 2^2 = -1$. It also makes the variance of $X - Y$ negative: $\text{Var}(X - Y) = -2$. I apologize for that.

Problem 4. Consider a coin with $P(H) = 40\%$.

(a) Suppose that you flip the coin 100 times and let X be the number of heads you get. Compute $E[X]$ and $\text{Var}(X)$.

Here X is binomial with parameters $n = 100$ and $p = 4/10$, so that

$$\begin{aligned} E[X] &= np = 100(4/10) = 40, \\ \text{Var}(X) &= npq = 100(4/10)(6/10) = 24. \end{aligned}$$

(b) Suppose that you flip the coin until you see heads for the first time, and let Y be the number of flips that you did. Compute $P(Y = 2)$ and $E[Y]$.

Here Y is geometric with parameter $p = 4/10$, so that

$$\begin{aligned} P(Y = k) &= pq^{k-1}, \\ P(Y = 2) &= pq = (4/10)(6/10) = 24/100, \\ E[Y] &= 1/p = 10/4. \end{aligned}$$

(c) Suppose that you flip the coin twice and define random variables X_1, X_2 by

$$X_i = \begin{cases} 1 & \text{if the } i\text{th flip is heads,} \\ 0 & \text{if the } i\text{th flip is tails.} \end{cases}$$

Compute $\text{Var}(X_1 + X_2)$.

Here X_1 and X_2 are Bernoulli with parameter $p = 4/10/$ so that

$$\text{Var}(X_1) = \text{Var}(X_2) = pq = (4/10)(6/10) = 24/100.$$

Then since X_1 and X_2 are independent we have

$$\text{Var}(X_1) + \text{Var}(X_2) = 24/100 + 24/100 = 48/100.$$

Alternatively, we can observe that $X_1 + X_2$ (the number of heads in two coin flips) is binomial with parameters $n = 2$ and $p = 4/10$, so that

$$\text{Var}(X_1 + X_2) = npq = 2(4/10)(6/10) = 48/100.$$

Problem 5. Suppose that an urn contains 2 red balls, 3 green balls and 1 blue ball. Grab 2 balls without replacement and consider the following random variables:

$$\begin{aligned} R &= \text{the number of red balls you get,} \\ G &= \text{the number of green balls you get.} \end{aligned}$$

(a) Write down formulas for the joint probabilities $P(R = k, G = \ell)$ and the marginal probabilities $P(R = k)$ and $P(G = \ell)$.

We use the formulas for hypergeometric probability:

$$P(R = k) = \binom{2}{k}\binom{4}{2-k} \Big/ \binom{6}{2},$$

$$P(G = \ell) = \binom{3}{\ell}\binom{3}{2-\ell} \Big/ \binom{6}{2},$$

$$P(R = k, G = \ell) = \binom{2}{k}\binom{3}{\ell}\binom{1}{2-k-\ell} \Big/ \binom{6}{2}.$$

(b) Fill in the joint pmf table, including the marginal probabilities:

To save space I will omit the denominator $\binom{6}{2} = 15$ from each cell.

$R \setminus G$	0	1	2	
0	0	3	3	6
1	2	6	0	8
2	1	0	0	1
	3	9	3	15

Note that the marginal probabilities are equal to the corresponding row and column sums, and that the probabilities sum to 1.