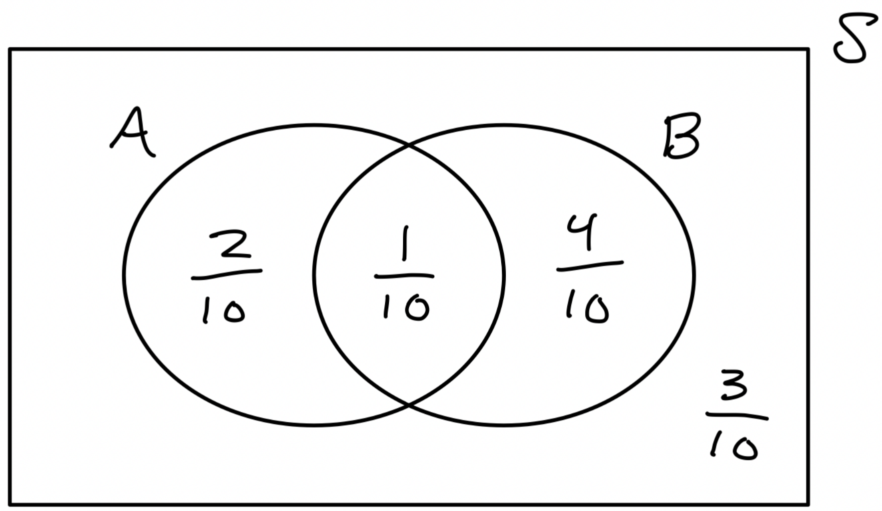
Math 224 **Exam 1**
Spring 2022 **Thurs Feb 17**

This is a closed book test. No electronic devices are allowed. There are 5 pages and 5 problems, each worth 6 points, for a total of 30 points.

Problem 1. Let S be the sample space of an experiment and consider two events $A, B \subseteq S$ with the following properties:

$$P(A \cap B) = 1/10, \qquad P(A \cap B') = 2/10 \qquad \text{and} \qquad P(A' \cap B) = 4/10.$$

(a) Label the probabilities of the four regions in the Venn diagram:



(b) Compute the probabilities $P(A)$ and $P(B)$.

We add up the probabilities in the corresponding regions of the diagram to get

$$P(A) = \frac{2}{10} + \frac{1}{10} = \frac{3}{10} \quad \text{and} \quad P(B) = \frac{1}{10} + \frac{4}{10} = \frac{5}{10}.$$

Alternatively, we could use the Law of Total Probability:

$$P(A) = P(A \cap B) + P(A \cap B'),$$
$$P(B) = P(A \cap B) + P(A' \cap B).$$

(c) Compute the probabilities $P(A|B)$ and $P(B|A)$.

Using part (a) and the definition of conditional probability gives

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/10}{5/10} = \frac{1}{5}$$

and

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/10}{3/10} = \frac{1}{3}.$$

Problem 2. A certain small country has car license plates consisting of 2 letters followed by 2 digits. Repetition of symbols is allowed. [There are 26 letters in the alphabet.]

(a) Calculate the number of possible license plates.

There are 26 possible letters and 10 possible digits. Since repetition is allowed, the number of possible license plates is

$$\underbrace{26}_{\text{1st letter}} \times \underbrace{26}_{\text{2nd letter}} \times \underbrace{10}_{\text{1st digit}} \times \underbrace{10}_{\text{1st digit}} = 26^2 \cdot 10^2 = 67600.$$

This is the size of the sample space for parts (b) and (c).

(b) What is the probability that a random license plate has no repeated symbols?

The number of license plates with no repeated symbols is

$$\underbrace{26}_{\text{1st letter}} \times \underbrace{25}_{\text{2nd letter}} \times \underbrace{10}_{\text{1st digit}} \times \underbrace{9}_{\text{1st digit}} = 58500.$$

Assuming that the license plates from part (a) are equally likely, the probability of getting no repeated symbols is

$$\frac{26 \cdot 25 \cdot 10 \cdot 9}{26 \cdot 26 \cdot 10 \cdot 10} = \frac{25 \cdot 9}{26 \cdot 10} = 86.5\%.$$

(c) What is the probability that a random license plate contains at least one vowel? [The vowels are $\{A, E, I, O, U\}$.]

Since there are 21 non-vowels, the number of license plates with **no vowels** is

$$\underbrace{21}_{\text{1st letter}} \times \underbrace{21}_{\text{2nd letter}} \times \underbrace{10}_{\text{1st digit}} \times \underbrace{10}_{\text{1st digit}} = 21^2 \cdot 10^2 = 44100.$$

Hence we have

$$\begin{aligned} P(\text{at least one vowel}) &= 1 - P(\text{no vowels}) \\ &= 1 - \frac{21^2 \cdot 10^2}{26^2 \cdot 10^2} \\ &= 1 - \left(\frac{21}{26}\right)^2 \\ &= 34.8\%. \end{aligned}$$

Problem 3. A fair 3-sided die has sides labeled a, b, c. Suppose that you roll the die 3 times and let A, B, C be the number of times you get a, b, c, respectively.

(a) Compute the probability of getting $A = 1$, $B = 0$ and $C = 2$.

For any non-negative integers i, j, k with $i + j + k = 3$, the general formula for multinomial probability gives

$$\begin{aligned} P(A = i, B = j, C = k) &= \frac{3!}{i!j!k!} \left(\frac{1}{3}\right)^i \left(\frac{1}{3}\right)^j \left(\frac{1}{3}\right)^k \\ &= \frac{3!}{i!j!k!} \left(\frac{1}{3}\right)^{i+j+k} \\ &= \frac{3!}{i!j!k!} \cdot \left(\frac{1}{3}\right)^3 \\ &= \frac{3!/(i!j!k!)}{27}. \end{aligned}$$

Since the die is fair, we can also think of the numerator $3!/(i!j!k!)$ as the number of outcomes with $A = i, B = j, C = k$ and the denominator $27 = 3^3$ as the total number of possible outcomes.

We were asked to compute

$$P(A = 1, B = 0, C = 2) = \frac{3!/(1!0!2!)}{27} = \frac{3}{27}.$$

Remark: The 3 in the numerator counts the corresponding outcomes: acc, aca, caa.

(b) Compute the probability of getting $A = B$.

There are two ways to get $A = B$; namely, $A = 0, B = 0, C = 3$ and $A = 1, B = 1, C = 1$. Thus we have

$$\begin{aligned} P(A = B) &= P(A = 0, B = 0, C = 3) + P(A = 1, B = 1, C = 1) \\ &= \frac{3!/(0!0!3!)}{27} + \frac{3!/(1!1!1!)}{27} \\ &= \frac{1}{27} + \frac{6}{27} \\ &= \frac{7}{27}. \end{aligned}$$

Remark: The 7 in the numerator counts the corresponding outcomes:

$$ccc, abc, acb, bac, bca, cab, cba.$$

(c) What is the probability of getting aca, in that order?

There is exactly one way for this to happen. Since the $27 = 3^3$ possible outcomes are equally likely, this gives

$$P(aca) = \frac{1}{27}.$$

We can also view this as

$$P(aca) = P(a)P(c)P(a) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3},$$

using the fact that the dice rolls are independent.

Problem 4. A standard deck of 52 cards contains 13 hearts, 13 diamonds, 13 spades and 13 clubs. The hearts and diamonds are "red cards" and the spaces and clubs are "black cards". Suppose that 5 cards are dealt at random (without replacement).

(a) Compute the probability of getting 1 heart, 1 diamond and 3 spades.

For any non-negative integers $h, d, s, c \geq 0$ with $h + d + s + c = 5$, the formula for hypergeometric probability gives

$$P(h \text{ hearts, } d \text{ diamonds, } s \text{ spades, } c \text{ clubs}) = \frac{\binom{13}{h}\binom{13}{d}\binom{13}{s}\binom{13}{c}}{\binom{52}{5}}$$

In our case we have

$$P(1 \text{ heart, } 1 \text{ diamond, } 3 \text{ spades}) = \frac{\binom{13}{1}\binom{13}{1}\binom{13}{3}\binom{13}{0}}{\binom{52}{5}} = 1.86\%.$$

Parts (b) and (c) use similar principles.

(b) Compute the probability of getting 2 red cards and 3 spades.

We can use the following shortcut:

$$P(2 \text{ red cards, } 3 \text{ spades}) = \frac{\binom{26}{2}\binom{13}{3}}{\binom{52}{5}} = 3.58\%.$$

We could also sum over the possible numbers of hearts and diamonds:

$$P(2 \text{ red, } 3 \text{ spades}) = \frac{\binom{13}{0}\binom{13}{2}\binom{13}{3}\binom{13}{0}}{\binom{52}{5}} + \frac{\binom{13}{1}\binom{13}{1}\binom{13}{3}\binom{13}{0}}{\binom{52}{5}} + \frac{\binom{13}{2}\binom{13}{0}\binom{13}{3}\binom{13}{0}}{\binom{52}{5}}$$

(c) Compute the probability of getting at least one red card.

This time we definitely want to use a short cut:

$$P(\text{at least one red}) = 1 - P(\text{all black}) = 1 - \frac{\binom{26}{0}\binom{26}{5}}{\binom{52}{5}} = 2.53\%.$$

Problem 5. An urn contains 2 red and 4 green balls. Two balls are drawn in order (and without replacement). Consider the following events:

$$A = \text{the first ball is red,}$$
$$B = \text{the second ball is green.}$$

(a) Compute the probabilities $P(A)$, $P(B)$ and $P(B|A)$.

This problem could be solved by considering a sample space S with $\#S = 6 \cdot 5 = 30$ and then counting the outcomes corresponding to A and B. But it is much easier to just ignore the other ball. The probability that the first ball is red (ignoring the second ball) is

$$P(A) = \frac{2}{2+4} = \frac{1}{3}.$$

The probability that the second ball is green (ignoring the first ball) is

$$P(B) = \frac{4}{2+4} = \frac{2}{3}.$$

If we **don't** ignore the first ball, then we get

$$P(B|A) = \frac{4}{1+4} = \frac{4}{5}.$$

(After one red ball is taken from the urn, there are 5 balls remaining and 4 of them are green.)

(b) Compute the probabilities $P(A \cap B)$ and $P(A \cup B)$.

From the definition of conditional probability we have

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{1}{3} \cdot \frac{4}{5} = \frac{4}{15}.$$

Then combining $P(A)$, $P(B)$ and $P(A \cap B)$ gives

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{2}{3} - \frac{4}{15} = \frac{11}{15}.$$

(c) Compute the probability $P(A|B)$.

Combining (a) and (b) with the definition of conditional probability gives

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{4/15}{2/3} = \frac{2}{5}.$$

Interpretation: Your friend takes two balls from the urn. Without knowing anything, you should believe that

$$P(\text{first ball is red}) = \frac{1}{3}.$$

If your friend then shows you that their second ball is green, you should update your belief to

$$P(\text{first ball is red, assuming second ball is green}) = \frac{2}{5}.$$