Math 224 **Spring 2018**
Homework 4 **Drew Armstrong**

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1 (St. Petersburg Paradox). I am running a lottery. I will let you flip a fair coin until you get heads. If the first head shows up on the k-th flip I will pay you r^k dollars.

(a) Compute your expected winnings when $r = 1$.
(b) Compute your expected winnings when $r = 1.5$.
(c) Compute your expected winnings when $r = 2$. Does this make any sense? How much would you be willing to pay me to play this game?

[Moral of the Story: The expected value is not always meaningful.]

2. I am running a lottery. I will sell 50 million tickets, 5 million of which will be winners.

(a) If you purchase 10 tickets, what is the probability of getting at least one winner?
(b) If you purchase 15 tickets, what is the probability of getting at least one winner?
(c) If you purchase n tickets, what is the probability of getting at least one winner?
(d) What is the smallest value of n such that your probability of getting a winner is greater than 50%? What is the smallest value of n that gives you a 95% chance of winning?

[Hint: If n is small, then each ticket is approximately a coin flip with $P(H) = 1/10$. In other words, for small values of n we have the approximation

$$\binom{45,000,000}{n} \Big/ \binom{50,000,000}{n} \approx (9/10)^n.]$$

3. Flip a fair coin 3 times and let

$$X = \text{“number of heads squared, minus the number of tails.”}$$

(a) Write down a table showing the pmf of X.
(b) Compute the expected value $\mu = E[X]$.
(c) Compute the variance $\sigma^2 = \mathrm{Var}(X)$.
(d) Draw the line graph of the pmf. Indicate the values of $\mu - \sigma, \mu, \mu + \sigma$ in your picture.

4. Let X and Y be random variables with supports $S_X = \{1, 2\}$ and $S_Y = \{1, 2, 3, 4\}$, and with joint pmf given by the formula

$$f_{XY}(k, \ell) = P(X = k, Y = \ell) = \frac{k + \ell}{32}.$$

(a) Draw the joint pmf table, showing the marginal probabilities in the margins.
(b) Compute the following probabilities directly from the table:

$$P(X > Y), \quad P(X \le Y), \quad P(Y = 2X), \quad P(X + Y > 3), \quad P(X + Y \le 3).$$

(c) Use the marginal distributions to compute $E[X], \mathrm{Var}(X)$ and $E[Y], \mathrm{Var}(Y)$.
(d) Use the table to compute the pmf of XY. Use this to compute $E[XY]$ and $\mathrm{Cov}(X, Y)$.
(e) Compute the correlation coefficient:

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y}.$$

Are the random variables X, Y independent? Why or why not?

5. Let X and Y be random variables with the following joint distribution:

$X \setminus Y$	-1	0	1
-1	0	0	$1/4$
0	$1/2$	0	0
1	0	0	$1/4$

(a) Compute the numbers $E[X], \mathrm{Var}(X)$ and $E[Y], \mathrm{Var}(Y)$.
(b) Compute the expected value $E[XY]$ and the covariance $\mathrm{Cov}(X, Y)$.
(c) Are the random variables X, Y independent? Why or why not?

[Moral of the Story: Uncorrelated does not always imply independent.]

6. Roll a fair 6-sided die twice. Let X be the number that shows up on the first roll and let Y be the number that shows up on the second roll. You may assume that X and Y are independent.

(a) Compute the covariance $\mathrm{Cov}(X, Y)$.
(b) Compute the covariance $\mathrm{Cov}(X, X + Y)$.
(c) Compute the covariance $\mathrm{Cov}(X, 2X + 3Y)$.

7. Let X_1 and X_2 be independent samples from a distribution with the following pmf:

k	0	1	2
$f(k)$	$1/4$	$1/2$	$1/4$

(a) Draw the joint pmf table of X_1 and X_2.
(b) Use your table to compute the pmf of $X_1 + X_2$.
(c) Compute the variance $\mathrm{Var}(X_1 + X_2)$ in two different ways.

8. Each box of a certain brand of cereal comes with a toy inside. If there are n possible toys and if the toys are distributed randomly, how many boxes do you expect to buy before you get them all?

(a) Assuming that you already have ℓ of the toys, let X_ℓ be the number of boxes you need to purchase until you get a new toy that you don't already have. Compute the expected value $E[X_\ell]$. [Hint: We can think of each new box purchased as a "coin flip" where $H =$"we get a new toy" and $T =$"we don't get a new toy." Thus X_ℓ is a geometric random variable. What is $P(H)$?]
(b) Let X be the number of boxes you purchase until you get all n toys. Thus we have
$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}.$$
Use part (a) and linearity to compute the expected value $E[X]$.
(c) Application: Suppose you continue to roll a fair 6-sided die until you see all six sides. How many rolls do you expect to make?