Math 224 **Spring 2018**
Homework 3 **Drew Armstrong**

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1. Consider a coin with $P(H) = p$ and $P(T) = q$. Flip the coin until the first head shows up and let X be the number of flips you made. The probability mass function and support of this *geometric random vabiable* are given by

$$P(X = k) = q^{k-1}p \qquad \text{and} \qquad S_X = \{1, 2, 3, \ldots\}.$$

(a) Use the geometric series $1 + q + q^2 + \cdots = (1 - q)^{-1}$ to show that

$$\sum_{k \in S_X} P(X = k) = 1.$$

(b) Differentiate the geometric series to get $0 + 1 + 2q + 3q^2 + \cdots = (1 - q)^{-2}$ and use this series to show that

$$E[X] = \sum_{k \in S_X} k \cdot P(X = k) = \frac{1}{p}.$$

(c) Application: Start rolling a fair 6-sided die. On average, how long do you have to wait until you see "1" for the first time?

Throughout I will assume that $0 < q = 1 - p < 1$. (The cases $p \in \{0, 1\}$ are slightly different.) For (a) we have

$$\begin{aligned}
\sum_{k \in S_X} P(X = k) &= \sum_{k=1}^{\infty} P(X = k) \\
&= \sum_{k=1}^{\infty} q^{k-1}p \\
&= p + qp + q^2p + q^3p + \cdots \\
&= p(1 + q + q^2 + q^3 + \cdots) \\
&= p/(1 - q) \\
&= p/p \\
&= 1.
\end{aligned}$$

In words: The total probability is 1. Then for (b) we have

$$\begin{aligned}
E[X] = \sum_{k \in S_X} k \cdot P(X = k) &= \sum_{k=1}^{\infty} k \cdot P(X = k) \\
&= \sum_{k=1}^{\infty} kq^{k-1}p \\
&= p + 2qp + 3q^2p + 4q^5p + \cdots \\
&= p(1 + 2q + 3q^2 + 4q^3 + \cdots) \\
&= p/(1 - q)^2 \\
&= p/p^2 \\
&= 1/p.
\end{aligned}$$

In words: We expect to see the first H on the $(1/p)$-th flip of the coin.
(c) Application: For example, we can think of a fair six-sided die as a strange coin where $H = \{\text{we get } 1\}$ and $T = \{\text{we don't get } 1\}$, so that $P(H) = p = 1/6$. Let X be the number of rolls until we see "1." Then by part (b) we have

$$E[\text{number of rolls until our first } 1] = E[X] = \frac{1}{p} = \frac{1}{1/6} = 6.$$

That makes sense.

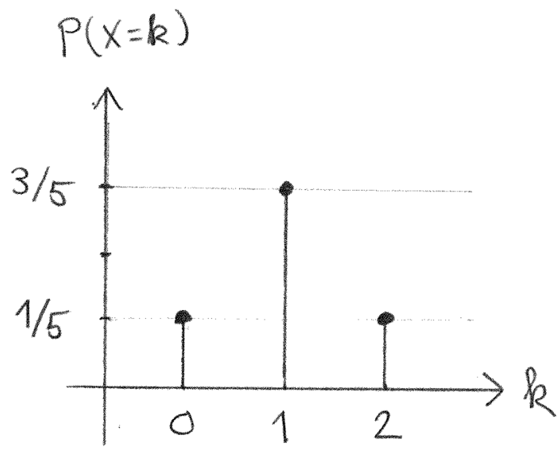
2. There are 2 red balls and 4 green balls in an urn. Suppose you grab 3 balls without replacement and let X be the number of red balls you get.

(a) What is the support of this random variable?
(b) Draw a picture of the probability mass function $f_X(k) = P(X = k)$.
(c) Compute the expected value $E[X]$. Does the answer make sense?

(a) The support is $S_X = \{0, 1, 2\}$.
(b) This X is a hypergeometric random variable with pmf given by the following table:

k	0	1	2
$P(X = k)$	$\frac{\binom{2}{0}\binom{4}{3}}{\binom{6}{3}} = \frac{4}{20}$	$\frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{12}{20}$	$\frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{4}{20}$

Here is the line graph:



(c) We have

$$E[X] = 0 \cdot P(X = 0) + 1 \cdot P(X = 2) + 2 \cdot P(X = 2) = 0 \cdot \frac{1}{5} + 1 \cdot \frac{3}{5} + 2 \cdot \frac{1}{5} = \frac{5}{5} = 1.$$

This answer makes sense for two reasons:

- The line graph is symmetric about $k = 1$, so $k = 1$ is the center of mass.
- One third of the balls in the urn are red. If we grab three balls then we expect one third of them (i.e., one ball) to be red.

3. Roll a pair of fair 6-sided dice and consider the following random variables:

$$X = \text{the number that shows up on the first roll},$$
$$Y = \text{the number that shows up on the second roll}.$$

(a) Write down all elements of the sample space S.
(b) Compute the probability mass function for the sum $f_{X+Y}(k) = P(X+Y=k)$ and draw the probability histogram.
(c) Compute the expected value $E[X+Y]$ in two different ways.
(d) Compute the probability mass function for the difference $f_{X-Y}(k) = P(X-Y=k)$ and draw the probability histogram.
(e) Compute the expected value $E[X-Y]$ in two different ways.
(f) Compute the probability mass function for the absolute value of the difference

$$f_{|X-Y|}(k) = P(|X-Y| = k)$$

and draw the probability histogram.
(g) Compute the expected value $E\left[|X-Y|\right]$. This time there is only one way to do it.

(a) I will assume that the dice are ordered.[1] Here is the sample space:

$$S = \left\{ \begin{array}{cccccc} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{array} \right\}$$

(b) To compute the probabilites $P(X+Y=k)$ we circle the events $\{X+Y=k\}$:

$$X+Y=$$

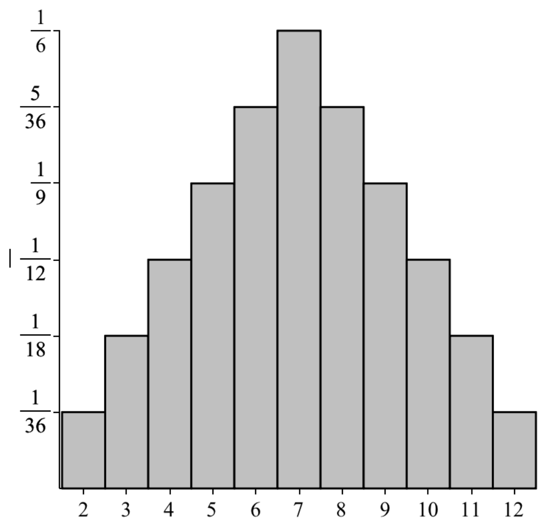
$$\begin{array}{ccccccc} 2 & 3 & 4 & 5 & 6 & 7 & \\ \{ 11 & 12 & 13 & 14 & 15 & 16 & 8 \\ 21 & 22 & 23 & 24 & 25 & 26 & 9 \\ 31 & 32 & 33 & 34 & 35 & 36 & 10 \\ 41 & 42 & 43 & 44 & 45 & 46 & 11 \\ 51 & 52 & 53 & 54 & 55 & 56 & 12 \\ 61 & 62 & 63 & 64 & 65 & 66 \, \} & \end{array}$$

[1]Unordered dice makes the problem much harder.

And then we count them up:

k	2	3	4	5	6	7	8	9	10	11	12
$P(X+Y=k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Here is the probability histogram:



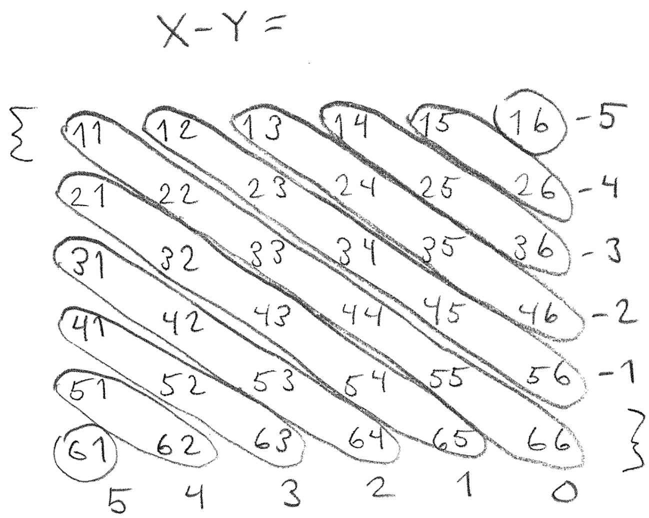
(c) We can compute the expected value directly from the formula:

$$E[X+Y] = \sum_{k=2}^{12} k \cdot P(X+Y=k)$$
$$= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} = \frac{252}{36} = 7.$$

Or we can use linearity. Recall from class that the average outcome in one roll of a fair die is $E[X] = E[Y] = 3.5$. Thus we have

$$E[X+Y] = E[X] + E[Y] = 3.5 + 3.5 = 7.$$

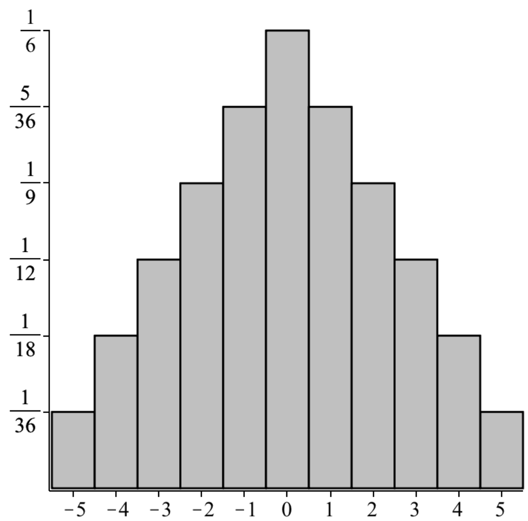
(d) To compute the probabilites $P(X-Y=k)$ we circle the events $\{X-Y=k\}$:



And then we count them up:

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
$P(X-Y=k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Here is the probability histogram:



Note that the histogram looks the same, just shifted 7 units to the left. That surprised me.

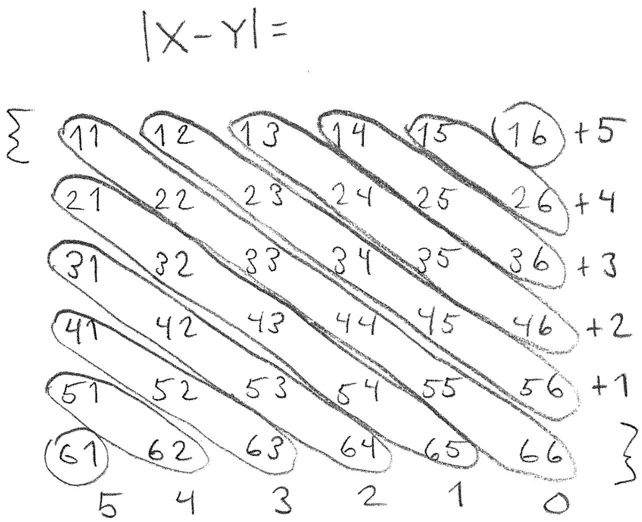
(e) We can compute the expected value directly from the formula:

$$
\begin{aligned}
E[X-Y] &= \sum_{k=-5}^{5} k \cdot P(X-Y=k) \\
&= -5 \cdot \frac{1}{36} - 4 \cdot \frac{2}{36} - \cdots + 4 \cdot \frac{2}{36} + 5 \cdot \frac{1}{36} = \frac{0}{36} = 0.
\end{aligned}
$$

Or we can use linearity:

$$E[X-Y] = E[X] - E[Y] = 3.5 - 3.5 = 0.$$

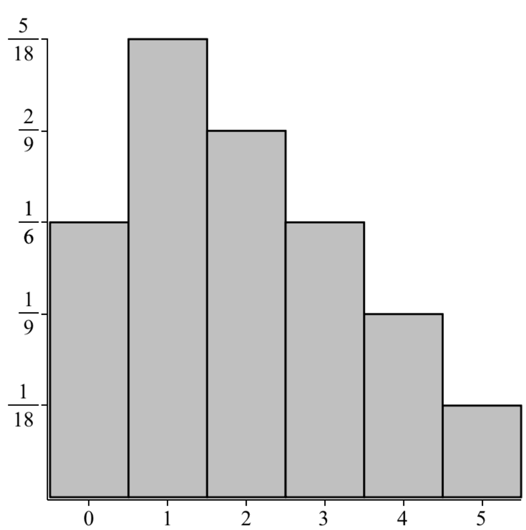
(f) When we compute $P(|X - Y| = k)$, some of the values from (d) turn positive:



Then we add them up:

k	0	1	2	3	4	5
$P(X - Y = k)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

And draw the histogram:



(g) We can compute the expected value directly from the formula:

$$E\left[|X - Y|\right] = \sum_{k=0}^{5} k \cdot P(|X - Y| = k)$$
$$= 0 \cdot \frac{6}{36} + 1 \cdot \frac{10}{36} + \cdots + 4 \cdot \frac{4}{36} + 5 \cdot \frac{2}{36} = \frac{70}{36} = 1.94.$$

This time the linearity of expectation doesn't help because the absolute value is not nice.

4. Let X be a random variable satisfying

$$E[X+1] = 3 \qquad \text{and} \qquad E[(X+1)^2] = 10.$$

Use this information to compute the following:

$$\mathrm{Var}(X+1), \quad E[X], \quad E[X^2] \quad \text{and} \quad \mathrm{Var}(X).$$

Solution: By definition we have

$$\mathrm{Var}(X+1) = E[(X+1)^2] - E[X+1]^2 = 10 - 3^2 = 1.$$

Then using linearity of expectation gives

$$\begin{aligned} E[X+1] &= 3 \\ E[X] + E[1] &= 3 \\ E[X] + 1 &= 3 \\ E[X] &= 2 \end{aligned}$$

and

$$\begin{aligned} E[(X+1)^2] &= 10 \\ E[X^2 + 2X + 1] &= 10 \\ E[X^2] + 2E[X] + 1 &= 10 \\ E[X^2] + 2(2) + 1 &= 10 \\ E[X^2] &= 5. \end{aligned}$$

Finally, we have

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = 5 - 2^2 = 1.$$

Note that this agrees with the general fact $\mathrm{Var}(X+\alpha) = \mathrm{Var}(X)$ when α is constant.

5. Let X be a random variable with mean $E[X] = \mu$ and variance $\mathrm{Var}(X) = \sigma^2 \neq 0$. Compute the mean and variance of the random variable Y defined by

$$Y = \frac{X - \mu}{\sigma}.$$

Solution: For the expected value we have

$$E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}E[X-\mu] = \frac{1}{\sigma}\left(E[X] - \mu\right) = \frac{1}{\sigma}(\mu - \mu) = 0.$$

For the variance we have

$$\begin{aligned} E[Y] &= E[Y^2] - \cancel{E[Y]^2} \\ &= E[Y^2] \\ &= E\left[\frac{(X-\mu)^2}{\sigma^2}\right] \\ &= \frac{1}{\sigma^2}E[(X-\mu)^2] \\ &= \frac{1}{\sigma^2}\mathrm{Var}(X) = \frac{1}{\sigma^2} \cdot \sigma^2 = 1. \end{aligned}$$

6. Let X be the number of strangers you must talk to until you find someone who shares your birthday. (Assume that each day of the year is equally likely and ignore February 29.)

(a) Find the probability mass function $P(X = k)$.
(b) Find the expected value $\mu = E[X]$.
(c) Find the *cumulative mass function* $P(X \le k)$. Hint: If X is a geometric random variable with pmf $P(X = k) = q^{k-1}p$, use the geometric series to show that

$$P(X \le k) = 1 - P(X > k) = 1 - \sum_{i=k+1}^{\infty} q^{i-1}p = 1 - q^k.$$

(d) Use part (c) to find the probability $P(\mu - 50 \le X \le \mu + 50)$ that X falls within ± 50 of the expected value. Hint:

$$P(\mu - 50 \le X \le \mu + 50) = P(X \le \mu + 50) - P(X \le \mu - 50 - 1).$$

(a) We can think of each stranger as a coin flip where "heads" means "they have the same birthday as you." Then X is a geometric random variable with $P(H) = p = 1/365$ and $P(T) = q = 1 - p = 364/365$. From Problem 1 we know that

$$P(X) = q^{k-1}p = \left(\frac{364}{365}\right)^{k-1}\left(\frac{1}{365}\right) = \frac{364^{k-1}}{365^k}.$$

(b) From Problem 1 we also know that

$$E[X] = \frac{1}{p} = \frac{1}{1/365} = 365.$$

That is, on average you will need to speak to 365 strangers until you find someone who shares your birthday. That makes sense.
(c) Note that $P(X \le k) = 1 - P(X > k)$ and

$$\begin{aligned} P(X > k) &= P(X = k+1) + P(X = k+2) + P(X = k+3) + \cdots \\ &= q^k p + q^{k+1}p + q^{k+3}p + \cdots \\ &= q^k p(1 + q + q^2 + \cdots) \\ &= q^k p \cdot \frac{1}{1-q} = q^k p \cdot \frac{1}{p} = q^k. \end{aligned}$$

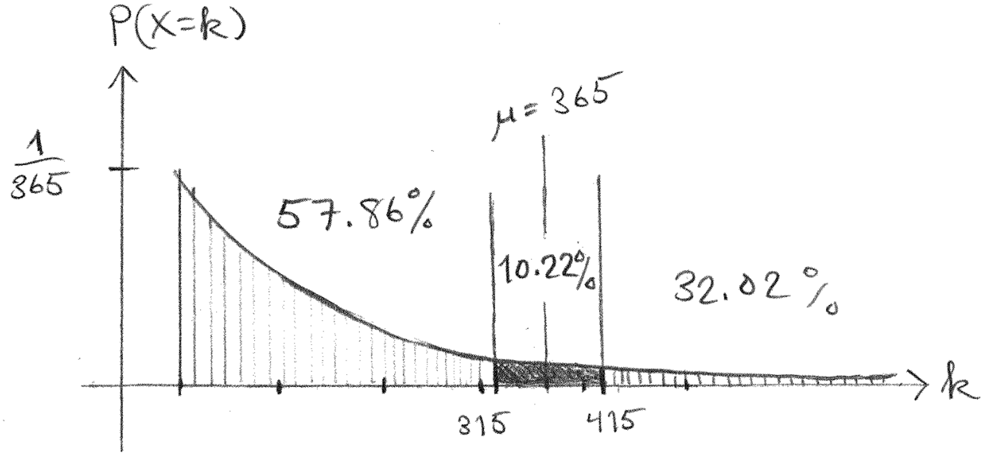
So we conclude that $P(X \le k) = 1 - q^k$.
(d) Continuing from part (c), we have for any whole numbers k and ℓ that

$$P(k \le X \le \ell) = P(X \le \ell) - P(X \le k - 1) = (1 - q^\ell) - (1 - q^{k-1}) = q^{k-1} - q^\ell.$$

In particular, we see that

$$P(315 \le X \le 415) = q^{314} - q^{415} = \left(\frac{364}{365}\right)^{314} - \left(\frac{364}{365}\right)^{415} = 10.22\%.$$

In other words, there is a 10.22% chance that you will need to ask between 315 and 415 people until you find someone who shares your birthday. Here is a picture of the pmf (not to scale, obviously):



7. I am running a lottery. I will sell 10 tickets, each for a price of \$1. The person who buys the winning ticket will receive a cash prize of \$5.

(a) If you buy one ticket, what is the expected value of your profit?
(b) If you buy two tickets, what is the expected value of your profit?
(c) If you buy n tickets ($0 \leq n \leq 10$), what is the expected value of your profit? Which value of n maximizes your expected profit?

[Remark: Profit equals prize money minus cost of the tickets.]

(a) Suppose you buy one ticket and let X be your profit. If you buy a losing ticket then your profit is $X = -1$ dollar and if you buy the winning ticket then your profit is $X = -1 + 5 = 4$ dollars. The probability of getting the winning ticket is $1/10$, so here is the pmf of X:

k	-1	$+4$
$P(X=k)$	$\frac{9}{10}$	$\frac{1}{10}$

Your expected profit is $E[X] = (-1)(9/10) + 4(1/10) = -5/10 = -0.5$ dollars.

(b) Let X be your profit from the purchase of two tickets. If both tickets are losers then $X = -2$ and if one ticket is a winner then $X = -2 + 5 = 3$. The number of ways to choose 2 out of 10 tickets is $\binom{10}{2} = 45$ and the number of way to choose 2 losing tickets out of 9 is $\binom{9}{2} = 36$. The number of ways to choose 1 winning ticket and 1 losing ticket is $\binom{1}{1}\binom{9}{1} = 9$. So here is the pmf of X:

k	-2	$+3$
$P(X=k)$	$\frac{\binom{1}{0}\binom{9}{2}}{\binom{10}{2}} = \frac{36}{45} = \frac{8}{10}$	$\frac{\binom{1}{1}\binom{9}{1}}{\binom{10}{2}} = \frac{9}{45} = \frac{2}{10}$.

Your expected profit is $E[X] = (-2)(8/10) + 3(2/10) = -10 - /10 = -1$ dollar.

(c) Let X be your profit from the purchase of n tickets (where $0 \leq n \leq 10$). Using a similar argument (and some algebraic manipulation) gives the pmf:

k	$-n$	$5-n$
$P(X=k)$	$\frac{\binom{1}{0}\binom{9}{n}}{\binom{10}{n}} = \frac{10-n}{10}$	$\frac{\binom{1}{1}\binom{9}{1}}{\binom{10}{2}} = \frac{n}{10}$.

Your expected profit is $E[X] = (-n)(10-n)/10 + (5-n)(n/10) = -n/2$ dollars. You will maximize your profit by purchasing $n = 0$ lottery tickets. But there is an easier way.

Easier Solution: Suppose you purchase n tickets and let X_i be your profit from the ith ticket. From part (a) we know that $E[X_i] = -0.5$. Now let $X = X_1 + X_2 + \cdots + X_n$ be your total profit. Using linearity of expectation gives

$$\begin{aligned} X &= X_1 + X_2 + \cdots + X_n \\ E[X] &= E[X_1 + X_2 + \cdots + X_n] \\ E[X] &= E[X_1] + E[X_2] + \cdots + E[X_n] \\ &= -0.5 - 0.5 - \cdots - 0.5 = n(-0.5) = -n/2. \end{aligned}$$

Note that linearity of expectation holds even though the random variables X_i are **not** independent of each other. That's pretty useful.

8. Consider a coin with $P(H) = p$ and $P(T) = q$. Flip the coin n times and let X be the number of heads you get. In this problem you will give a bad proof that $E[X] = np$.

(a) Use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(b) Complete the following computation:

$$\begin{aligned} E[X] &= \sum_{k=0}^{n} k \cdot P(X=k) \\ &= \sum_{k=1}^{n} k \cdot P(X=k) \\ &= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} \\ &= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} \\ &= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k q^{n-k} \\ &= \cdots \end{aligned}$$

(a) Note that

$$n\binom{n-1}{k-1} = n \cdot \frac{(n-1)!}{(k-1)!\,[(n-\cancel{1}) - (k-\cancel{1})]!} = \frac{n(n-1)!}{(k-1)(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

and

$$k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \cancel{k} \cdot \frac{n!}{\cancel{k}(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}.$$

Alternate Proof: Suppose you want to choose a k-member club from a classroom of n students, where one member of the club will serve as president. On the one hand, you can choose the club members in $\binom{n}{k}$ ways. Then there are k ways to choose the president. On the other hand, you can first choose a student to serve as club president. There are n ways to do this. Then there are $\binom{n-1}{k-1}$ ways to choose the remaining $k-1$ club members from the remaining $n-1$ students.

(b) Since $p+q=1$, let me first note that

$$1 = 1^{n-1} = (p+q)^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} p^{\ell} q^{(n-1)-\ell}.$$

Then making the substitution $k = \ell + 1$ gives

$$\begin{aligned}
E[X] &= \cdots \\
&= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k q^{n-k} \\
&= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k q^{n-k} \\
&= np \left(\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-k} \right) \\
&= np \left(\sum_{\ell=0}^{n-1} \binom{n-1}{(\ell+1)-1} p^{(\ell+1)-1} q^{n-(\ell+1)} \right) \\
&= np \left(\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} p^{\ell} q^{(n-1)-\ell} \right) \\
&= np(1) \\
&= np,
\end{aligned}$$

as desired.

Remark: That was certainly a bad way to solve the problem.