Math 224 **Spring 2018**
Homework 2 **Drew Armstrong**

1. Suppose that a fair s-sided die is rolled n times.

(a) If the i-th side is labeled a_i then we can think of the sample space S as the set of all words of length n from the alphabet $\{a_1, \ldots, a_s\}$. Find $\#S$.

(b) Let E be the event that "the 1st side shows up k_1 times, and ... and the s-th side shows up k_s times. Find $\#E$. [Hint: The elements of E are words of length n in which the letter a_i appears k_i times.]

(c) Compute the probability $P(E)$. [Hint: Since the die is fair you can assume that the outcomes in S are equally likely.]

The purpose of this problem was to encourage you to look at the course notes.

(a) To form a word of length n from the alphabet $\{a_1, \ldots, a_s\}$, there are s choices for the 1st letter, then s choices for the 2nd letter, then ... then s choices for the nth letter. Hence

$$\#S = \#(\text{words}) = \underbrace{s}_{\text{1st letter}} \times \underbrace{s}_{\text{2nd letter}} \times \cdots \times \underbrace{s}_{n\text{th letter}} = s^n.$$

(b) Note that $k_1 + k_2 + \cdots + k_s = n$. To form a word from k_1 copies of a_1, k_2 copies of a_2, ... and k_s copies of a_s, we can first form a permutation of the n symbols in $n!$ ways, but then we have to divide by $k_i!$ for each i to remove the orderings between the indistinguishable copies of a_i. Hence

$$\#E = \binom{n}{k_1, k_2, \ldots, k_s} = \frac{n!}{k_1!k_2!\cdots k_s!}.$$

(c) Since all outcomes are equally likely we have

$$P(E) = \frac{\#E}{\#S} = \binom{n}{k_1, k_2, \ldots, k_s} \cdot \frac{1}{s^n}.$$

2. In a certain state lottery four numbers are drawn (one and at a time and with replacement) from the set $\{1, 2, 3, 4, 5, 6\}$. You win if any permutation of your selected numbers is drawn. Rank the following selections in order of how likely each is to win.

(a) You select $1, 2, 3, 4$.
(b) You select $1, 3, 3, 5$.
(c) You select $4, 4, 6, 6$.
(d) You select $3, 5, 5, 5$.
(e) You select $4, 4, 4, 4$.

One way to solve this is to view it as an application of Problem 1. The problem is equivalent to rolling a fair $s = 6$-die $n = 4$ times, where the sides are labeled $\{1, 2, 3, 4, 5, 6\}$. The sample space has size

$$\#S = 6^4 = 1296.$$

In general, the number of outcomes in which 1 appears k_1 times, 2 appears k_2 times ... and 6 appears k_6 times is given by

$$\binom{4}{k_1, k_2, k_3, k_4, k_5, k_6} = \frac{4!}{k_1!k_2!k_3!k_4!k_5!k_6!}.$$

(a) The number of permutations of $1, 2, 3, 4$ is

$$\binom{4}{1,1,1,1,0,0} = \frac{4!}{1!1!1!1!0!0!} = 24,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{24}{1296} = 1.85\%.$$

(b) The number of permutations of $1, 3, 3, 5$ is

$$\binom{4}{1,0,2,0,1,0} = \frac{4!}{1!0!2!0!1!0!} = 12,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{12}{1296} = 0.93\%.$$

(c) The number of permutations of $4, 4, 6, 6$ is

$$\binom{4}{0,0,0,2,0,2} = \frac{4!}{0!0!0!2!0!2!} = 6,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{6}{1296} = 0.46\%.$$

(d) The number of permutations of $3, 5, 5, 5$ is

$$\binom{4}{0,0,1,0,3,0} = \frac{4!}{0!0!1!0!3!0!} = 4,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{4}{1296} = 0.31\%.$$

(e) The number of permutations of $4, 4, 4, 4$ is

$$\binom{4}{0,0,0,4,0,0} = \frac{4!}{0!0!0!4!0!0!} = 1,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{1}{1296} = 0.077\%.$$

Hence the order of winning is

$$\text{(a)} > \text{(b)} > \text{(c)} > \text{(d)} > \text{(e)}.$$

3. A bridge hand consists of 13 (unordered) cards taken (at random and without replacement) from a standard deck of 52. Recall that a standard deck contains 13 hearts and 13 diamonds (which are red cards), 13 clubs and 13 spades (which ard black cards). Find the probabilities of the following hands.

(a) 4 hearts, 3 diamonds, 2 spades and 4 clubs.
(b) 4 hearts, 3 diamonds and 6 black cards.
(c) 7 red cards and 6 black cards.

Let S be the set of possible bridge hands. The number of ways to choose a bridge hand (13 unordered cards without replacement from a deck of 52) is approximately 635 billion:

$$\#S = \binom{52}{13} = \frac{52!}{13!49!} = 635,013,559,600.$$

(a) The number of ways to choose 4 hearts, 3 diamonds, 2 spades and 4 clubs is

$$\underbrace{\binom{13}{4}}_{\substack{\text{choose}\\ \text{hearts}}} \times \underbrace{\binom{13}{3}}_{\substack{\text{choose}\\ \text{diamonds}}} \times \underbrace{\binom{13}{2}}_{\substack{\text{choose}\\ \text{spades}}} \times \underbrace{\binom{13}{4}}_{\substack{\text{choose}\\ \text{clubs}}} = 11,404,407,300,$$

hence the probability is

$$P(4\heartsuit\text{'s}, 3\diamondsuit\text{'s}, 2\spadesuit\text{'s}, 4\clubsuit\text{'s}) = \frac{\binom{13}{4}\binom{13}{3}\binom{13}{2}\binom{13}{4}}{\binom{52}{13}} = 1.8\%.$$

(a) The number of ways to choose 4 hearts, 3 diamonds and 6 black cards

$$\underbrace{\binom{13}{4}}_{\substack{\text{choose}\\ \text{hearts}}} \times \underbrace{\binom{13}{3}}_{\substack{\text{choose}\\ \text{diamonds}}} \times \underbrace{\binom{26}{6}}_{\substack{\text{choose}\\ \text{black cards}}} = 47,079,732,700,$$

hence the probability is

$$P(4\heartsuit\text{'s}, 3\diamondsuit\text{'s}, 6 \text{ black}) = \frac{\binom{13}{4}\binom{13}{3}\binom{26}{6}}{\binom{52}{13}} = 7.4\%.$$

(a) The number of ways to choose 7 red cards and 6 black cards is

$$\underbrace{\binom{26}{7}}_{\substack{\text{choose}\\ \text{red cards}}} \times \underbrace{\binom{26}{6}}_{\substack{\text{choose}\\ \text{black cards}}} = 151,445,294,000,$$

hence the probability is

$$P(7 \text{ red}, 6 \text{ black}) = \frac{\binom{26}{7}\binom{26}{6}}{\binom{52}{13}} = 23.8\%.$$

4. Two cards are drawn (in order and without replacement) from a standard deck of 52. Consider the events

$$A = \{\text{the first card is a heart}\}$$
$$B = \{\text{the second card is red}\}.$$

Compute the probabilities

$$P(A), \quad P(B), \quad P(B|A), \quad P(A \cap B), \quad P(A|B).$$

Since $13/52 = 1/4$ of the cards are hearts we have $P(A) = 1/4$. Similarly, since $26/52 = 1/2$ of the cards are red we have $P(B) = 1/2$.[1] If the first card is chosen to be a heart then 25 of the remaining 51 card are red, hence

$$P(B|A) = \frac{25}{51}.$$

From this we can compute

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{1}{4} \cdot \frac{25}{51} = \frac{25}{204},$$

and, finally, Bayes' Theorem gives

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)} = \left(\frac{1}{4} \cdot \frac{25}{51}\right) / \left(\frac{1}{2}\right) = \frac{25}{102}.$$

Since $P(A|B) < P(A)$ and $P(B|A) < P(B)$ we say that A and B are *negatively correlated*.

5. An urn contains 2 red and 2 green balls. Your friend selects two balls (at random and without replacement) and tells you that at least one of the balls is red. What is the probability that the other ball is also red?

First Solution. Let the two balls be ordered so that $\#S = 4 \times 3 = 12$. Consider the events

$$A = \{\text{1st ball is red}\},$$
$$B = \{\text{2nd ball is red}\}.$$

Observe that we have $P(A) = 1/2$, $P(B) = 1/2$ and $P(B|A) = 1/3$ (if the first ball is red then 1 of the 3 remaining balls is red). Note also that we have $(A \cap B) \cap (A \cup B) = A \cap B$. We are looking for the probability

$$\begin{aligned} &P(\text{ both balls are red } | \text{ at least one ball is red }) \\ &= P(A \cap B | A \cup B) \\ &= \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} \\ &= \frac{P(A \cap B)}{P(A \cup B)}. \end{aligned}$$

In order to compute this we observe that

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

and

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{6} = \frac{5}{6},$$

so that

$$P(A \cap B | A \cup B) = \frac{P(A \cap B)}{P(A \cup B)} = \frac{1/6}{5/6} = \frac{1}{5}.$$

Second Solution. Let the two balls be unordered so that $\#S = \binom{4}{2} = 6$. Let X be the number of red balls we get, so that

$$P(X = k) = \frac{\binom{2}{k}\binom{2}{2-k}}{\binom{4}{2}}.$$

[1] To see this we just ignore the first card. There are harder ways to compute $P(B)$ but you will always get the same answer.

We are looking for the probability

$$\begin{aligned}
P(X = 2|X \geq 1) &= \frac{P(X = 2 \text{ and } X \geq 1)}{P(X \geq 1)} \\
&= \frac{P(X = 2)}{P(X \geq 1)} \\
&= \frac{P(X = 2)}{1 - P(X = 0)} \\
&= \frac{\binom{2}{2}\binom{2}{0}/\binom{4}{2}}{1 - \binom{2}{0}\binom{2}{2}/\binom{4}{2}} \\
&= \frac{1/6}{1 - 1/6} \\
&= \frac{1}{5}.
\end{aligned}$$

6. There are two bowls on a table. The first bowl contains 3 red chips and 3 green chips. The second bowl contains 2 red chips and 4 green chips. Your friend walks up to the table and chooses one chip at random. Consider the events

$$\begin{aligned}
B_1 &= \{\text{the chip comes from the first bowl}\}, \\
B_2 &= \{\text{the chip comes from the second bowl}\}, \\
R &= \{\text{the chip is red}\}.
\end{aligned}$$

(a) Compute the probabilities $P(R|B_1)$ and $P(R|B_2)$.
(b) Assuming that your friend is equally likely to choose either bowl (i.e., $P(B_1) = P(B_2) = 1/2$), compute the probability $P(R)$ that the chip is red.
(c) Compute $P(B_1|R)$. That is, assuming that your friend chose a red chip, what is the probability that they got it from the first bowl?

(a) We have $P(R|B_1) = \frac{3}{3+3} = \frac{1}{2}$ and $P(R|B_2) = \frac{2}{2+4} = \frac{1}{3}$.

(b) Since the bowls are equally likely, we might as well dump all the chips into one bowl. Then we have

$$P(R) = \frac{2+3}{6+6} = \frac{5}{12}.$$

Alternatively, the Law of Total Probability gives

$$\begin{aligned}
R &= (R \cap B_1) \sqcup (R \cap B_2) \\
P(R) &= P(R \cap B_1) + P(R \cap B_2) \\
P(R) &= P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2) \\
&= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{12}.
\end{aligned}$$

(c) Bayes' Theorem gives

$$P(B_1|R) = \frac{P(B_1) \cdot P(R|B_1)}{P(R)} = \frac{(1/2)(1/2)}{5/12} = \frac{3}{5} = 60\%.$$

[Remark: Before we see the chip, there is a 50% chance that it came from the first bowl. After we see that the chip is red, there is a 60% chance that it came from the first bowl.]

7. A diagnostic test is administered to a random person to determine if they have a certain disease. Consider the events

$$T = \{\text{the test returns positive}\},$$
$$D = \{\text{the person has the disease}\}.$$

Suppose that the test has the following "false positive" and "false negative" rates:

$$P(T|D') = 2\% \quad \text{and} \quad P(T'|D) = 3\%.$$

(a) For any events A, B recall that the Law of Total Probability says

$$P(A) = P(A \cap B) + P(A \cap B').$$

Use this to prove that

$$1 = P(B|A) + P(B'|A).$$

(b) Use part (a) to compute the probability $P(T|D)$ of a "true positive" and the probability $P(T'|D')$ of a "true negative."

(c) Assume that 10% of the population has the disease, so that $P(D) = 10\%$. In this case compute the probability $P(T)$ that a random person tests positive. [Hint: The Law of Total Probability says $P(T) = P(T \cap D) + P(T \cap D')$.]

(d) Suppose that a random person is tested and the test returns positive. Compute the probability $P(D|T)$ that this person actually has the disease. Is this a good test?

(a) We divide both sides by $P(A)$ to get

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B') \\ \frac{P(A)}{P(A)} &= \frac{P(A \cap B)}{P(A)} + \frac{P(A \cap B')}{P(A)} \\ 1 &= P(B|A) + P(B'|A). \end{aligned}$$

(b) We have

$$P(T|D) = 1 - P(T'|D) = 97\% \qquad \text{and} \qquad P(T'|D') = 1 - P(T|D') = 98\%.$$

(c) The Law of Total Probability gives

$$\begin{aligned} T &= (T \cap D) \sqcup (T \cap D') \\ P(T) &= P(T \cap D) + P(T \cap D') \\ &= P(D) \cdot P(T|D) + P(D') \cdot P(T|D') \\ &= (0.1)(0.97) + (0.9)(0.02) \\ &= 11.5\%. \end{aligned}$$

(d) Bayes' Theorem gives

$$\begin{aligned} P(D|T) &= \frac{P(D \cap T)}{P(T)} \\ &= \frac{P(D) \cdot P(T|D)}{P(D) \cdot P(T|D) + P(D') \cdot P(T|D')} \\ &= \frac{(0.1)(0.97)}{(0.1)(0.97) + (0.9)(0.02)} \\ &= 87\%. \end{aligned}$$

This doesn't seem like a very good test to me.

8. Consider a classroom containing n students. We ask each student for their birthday, which we record as a number from the set $\{1, 2, \ldots, 365\}$ (i.e., we ignore leap years). Let S be the sample space.

(a) Explain why $\#S = 365^n$.
(b) Let E be the event that {no two students have the same birthday}. Compute $\#E$.
(c) Assuming that all birthdays are equally likely, compute the probability of the event

$$E' = \{\text{at least two students have the same birthday}\}.$$

(d) Find the smallest value of n such that $P(E') > 50\%$.

(a) Suppose that the students are ordered (say, alphabetically by last name). Then we have

$$\#S = \underbrace{365}_{\substack{\text{1st student's}\\ \text{birthday}}} \times \underbrace{365}_{\substack{\text{2nd student's}\\ \text{birthday}}} \times \cdots \times \underbrace{365}_{\substack{n\text{th student's}\\ \text{birthday}}} = 365^n.$$

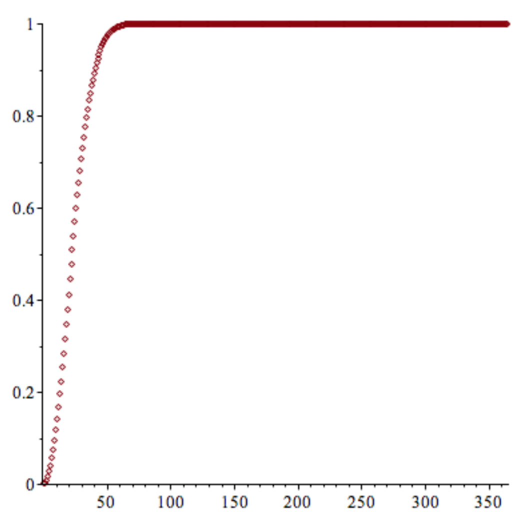
(b) If no two students are allowed to have the same birthday then for $n \geq 366$ we have $\#E = 0$ and for $n \leq 365$ we have

$$\#E = \underbrace{365}_{\substack{\text{1st student's}\\ \text{birthday}}} \times \underbrace{364}_{\substack{\text{2nd student's}\\ \text{birthday}}} \times \cdots \times \underbrace{(365 - n + 1)}_{\substack{n\text{th student's}\\ \text{birthday}}} = \frac{365!}{(365 - n)!}.$$

(c) If $n \geq 366$ we have $P(E) = 0$ and hence $P(E') = 1 - P(E) = 1$. If $n \leq 365$ then assuming all birthdays are equally likely gives

$$\begin{aligned} P(\text{at least two share a birthday}) &= 1 - P(\text{no two share a birthday}) \\ P(E') &= 1 - P(E) \\ &= 1 - \frac{\#E}{\#S} \\ &= 1 - \frac{365!/(365 - n)!}{365^n}. \end{aligned}$$

(d) Here is a plot of the probabilites $P(E')$ for values of n from 1 to 365. Note that the probability rises from 0% when $n = 1$ to 100% when $n = 366$.



At some point the probability must cross 50% and it seems from the diagram that this happens around $n = 25$. To be precise, I used my computer to find the following:

- For $n = 22$ students, the probability that at least two share a birthday is
$$1 - P(E) = 1 - \frac{365!/(365-22)!}{365^{22}} = 47.57\%.$$
- For $n = 23$ students, the probability that at least two share a birthday is
$$1 - P(E) = 1 - \frac{365!/(365-23)!}{365^{23}} = 50.73\%.$$

Do you find the number 23 surprisingly small? That's why this problem is sometimes also called the **birthday paradox**.[2]

9. It was not easy to find a formula for the entries of Pascal's Triangle. However, once we've found the formula it is not difficult to check that the formula is correct.

(a) Explain why $n! = n \times (n-1)!$.
(b) Use part (a) and the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to prove that
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Part (a) is self-explanatory. For part (b) we use part (a) to get a common denominator:
$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} + \frac{(n-1)!}{k!\,[(n-1)-k]!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \left(\frac{k}{k}\right) \cdot \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \cdot \left(\frac{n-k}{n-k}\right) \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!}
\end{aligned}$$

Then we use part (a) to simplify the numerator:
$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{k(n-1)! + (n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{[\cancel{k} + (n-\cancel{k})] \cdot (n-1)!}{k!(n-k)!} \\
&= \frac{n \cdot (n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

[2] In our class of $n = 35$ students there is an 81.4% chance that at least two students share a birthday.