Math 224 **Spring 2018**
Homework 1 **Drew Armstrong**

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1. Suppose that a fair coin is flipped 6 times in sequence and let X be the number of "heads" that show up. Draw Pascal's triangle down to the sixth row (recall that the zeroth row consists of a single 1) and use your table to compute the probabilities $P(X = k)$ for $k = 0, 1, 2, 3, 4, 5, 6$.

2. Suppose that a fair coin is flipped 4 times in sequence.

(a) List all 16 outcomes in the sample space S.

(b) List the outcomes in each of the following events:

$A = \{\text{at least 3 heads}\}$,
$B = \{\text{at most 2 heads}\}$,
$C = \{\text{heads on the 2nd flip}\}$,
$D = \{\text{exactly 2 tails}\}$.

(c) Assuming that all outcomes are **equally likely**, use the formula $P(E) = \#E/\#S$ to compute the following probabilities:

$$P(A \cup B), \quad P(A \cap B), \quad P(C), \quad P(D), \quad P(C \cap D).$$

3. Draw Venn diagrams to verify *de Morgan's laws*: For all events $E, F \subseteq S$ we have

(a) $(E \cup F)' = E' \cap F'$,

(b) $(E \cap F)' = E' \cup F'$.

4. Suppose that a fair coin is flipped until heads appears. The sample space is

$$S = \{H, TH, TTH, TTTH, TTTTH, \ldots\}.$$

However these outcomes are **not equally likely**.

(a) Let E_k be the event $\{\text{first } H \text{ occurs on the } k\text{th flip}\}$. Explain why $P(E_k) = 1/2^k$. [Hint: The outcomes of the coin flips are **independent**.]

(b) Use the "geometric series" to verify that the sum of all the probabilities equals 1:

$$\sum_{k=1}^{\infty} P(E_k) = 1.$$

5. Suppose that $P(A) = 0.5$, $P(B) = 0.6$ and $P(A \cap B) = 0.3$. Use this information to compute the following probabilities. A Venn diagram may be helpful.

(a) $P(A \cup B)$,

(b) $P(A \cap B')$,

(c) $P(A' \cup B')$.

6. Let X be a real number that is selected randomly from $[0, 1]$, i.e., the closed interval from zero to one. Use your intuition to assign values to the following probabilities:

(a) $P(X = 1/2)$,

(b) $P(0 \leq X \leq 1/2)$,

(c) $P(0 < X < 1/2)$,

(d) $P(1/3 < X \leq 3/4)$,

(e) $P(-1 < X < 3/4)$.

7. Consider a strange coin with $P(H) = p$ and $P(T) = q = 1 - p$. Suppose that you flip the coin n times and let X be the number of heads that you get. Find a formula for the probability $P(X \geq 1)$. [Hint: Observe that $P(X \geq 1) + P(X = 0) = 1$. Maybe it's easier to find a formula for $P(X = 0)$.]

8. Suppose that you roll a pair of fair six-sided dice.

(a) Write down all elements of the sample space S. What is $\#S$? Are the outcomes equally likely? [Hopefully, yes.]

(b) Compute the probability of getting a "double six." [Hint: Let $E \subseteq S$ be the subset of outcomes that correspond to getting a "double six." Assuming that the outcomes of your sample space are equally likely, you can use the formula $P(E) = \#E/\#S$.]

9. Analyze the Chevalier de Méré's two experiments:

(a) Roll a fair six-sided die 4 times and let X be the number of "sixes" that you get. Compute $P(X \geq 1)$. [Hint: You can think of a die roll as a "strange coin flip," where $H =$"six" and $T =$"not six." Use Problem 7.]

(b) Roll a pair of fair six-sided dice 24 times and let Y be the number of "double sixes" that you get. Compute $P(Y \geq 1)$. [Hint: You can think of rolling two dice as a "very strange coin flip," where $H =$"double six" and $T =$"not double six." Use Problems 7 and 8.]

10. Roll a fair six-sided die three times in sequence, and consider the events

$$\begin{aligned} E_1 &= \{\text{you get 1 or 2 or 3 on the first roll}\}, \\ E_2 &= \{\text{you get 1 or 3 or 5 on the second roll}\}, \\ E_3 &= \{\text{you get 2 or 4 or 6 on the third roll}\}. \end{aligned}$$

You can assume that $P(E_1) = P(E_2) = P(E_3) = 1/2$.

(a) Explain why $P(E_1 \cap E_2) = P(E_1 \cap E_3) = P(E_2 \cap E_3) = 1/4$ and $P(E_1 \cap E_2 \cap E_3) = 1/8$.

(b) Use this information to compute $P(E_1 \cup E_2 \cup E_3)$.