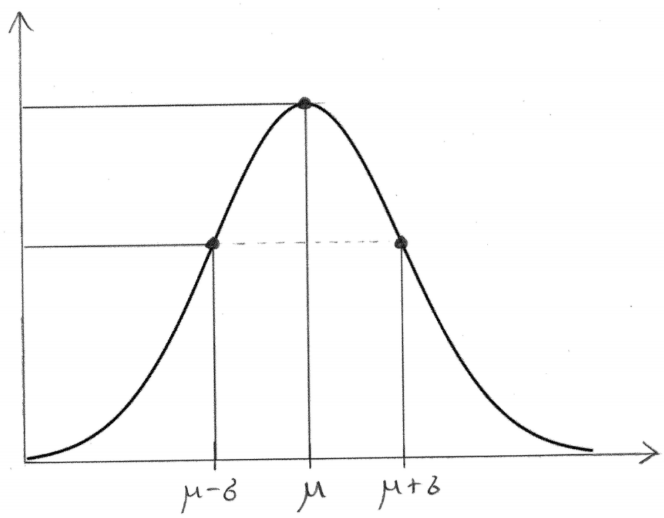
Version A

Problem 1. Let X be a **normal** random variable with mean μ and variance σ^2.

(a) Tell me the probability density function f_X.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

(b) Sketch the graph of f_X, showing the maximum and the points of inflection.



(c) Compute the probability $P(\mu + \sigma < X < \mu + 2\sigma)$.

Since $(X - \mu)/\sigma$ is standard normal, we have

$$\begin{aligned} P(\mu + \sigma < X < \mu + 2\sigma) &= P(\sigma < X - \mu < 2\sigma) \\ &= P\left(1 < \frac{X - \mu}{\sigma} < 2\right) \\ &= \Phi(2) - \Phi(1) \\ &= 0.9772 - 0.8413 = \boxed{13.59\%.} \end{aligned}$$

Problem 2. Let X be a continuous random variable with the following pdf:

$$f_X(x) = \begin{cases} \frac{3}{4}(1 - x^2) & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $\mu = E[X]$ and $\sigma^2 = \text{Var}(X)$.

We have

$$\mu = E[X] = \int_{-1}^{1} x \cdot \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{-1}^{1} = \boxed{0}$$

and

$$E[X^2] = \int_{-1}^{1} x^2 \cdot \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_{-1}^{1} = \frac{1}{5},$$

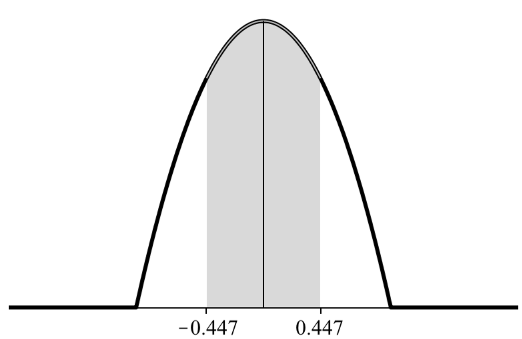
hence $\sigma^2 = \text{Var}(X) = E[X^2] - \mu^2 = \boxed{1/5.}$

(b) Compute the probability $P(\mu - \sigma < X < \mu + \sigma)$.

Since $\mu = 0$ and $\sigma = \sqrt{1/5} = 0.447$ we have

$$\begin{aligned} P(\mu - \sigma < X < \mu + \sigma) &= P(-0.447 < X < 0.447) \\ &= \int_{-0.447}^{0.447} \frac{3}{4}(1-x^2)\,dx \\ &= \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Bigg|_{-0.447}^{0.447} = \boxed{62.6\%.} \end{aligned}$$

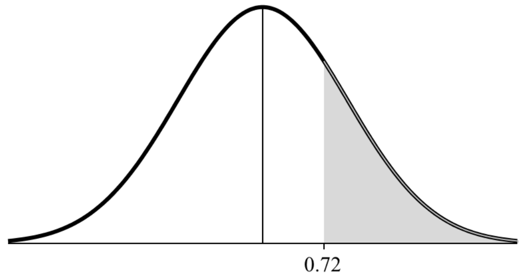
(c) Draw the graph of f_X, showing the region whose area you computed in part (b).



Problem 3. Let $Z \sim N(0,1)$ be a standard normal random variable.

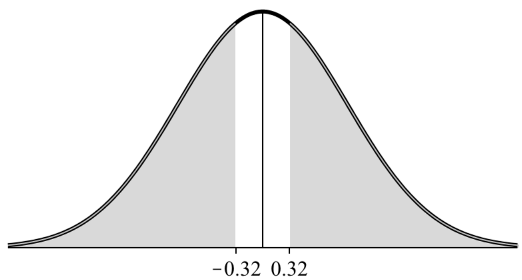
(a) Find α such that $P(Z > 0.72) = \alpha$ and draw a picture to illustrate your answer.

The area of the shaded region is $\alpha = 1 - \Phi(0.72) = 1 - 0.7642 = 0.2358$:



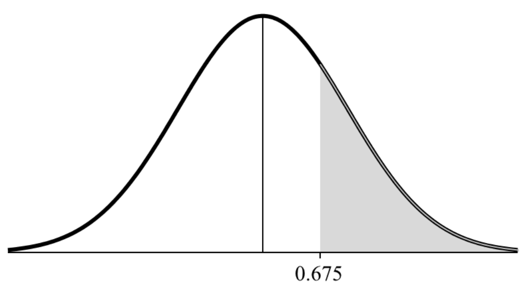
(b) Find c such that $P(|Z| > c) = 0.75$ and draw a picture to illustrate your answer.

We have $c = 0.32$. The area of the shaded region is 0.75:



(c) Find d such that $P(Z > d) = 0.25$ and draw a picture to illustrate your answer.

We have $d = 0.675$. The area of the shaded region is 0.25:



Problem 4. Let $X_1, X_2, \ldots, X_{20}$ be an iid sequence of random variables with

$$\mu = E[X_i] = 6 \qquad \text{and} \qquad \sigma^2 = \text{Var}(X_i) = 4.$$

Consider the sum X and the average $\overline{X}$ of these random variables:

$$X = X_1 + X_2 + \cdots + X_{20} \qquad \text{and} \qquad \overline{X} = \frac{1}{20} \cdot X.$$

(a) Compute the numbers $E[X]$, $\text{Var}(X)$, $E[\overline{X}]$ and $\text{Var}(\overline{X})$.

$$\begin{aligned} E[X] &= 20 \cdot \mu = \boxed{120,} \\ \text{Var}(X) &= 20 \cdot \sigma^2 = \boxed{80,} \\ E[\overline{X}] &= \mu = \boxed{6,} \\ \text{Var}(\overline{X}) &= \sigma^2/20 = 4/20 = \boxed{1/5.} \end{aligned}$$

(b) Use the Central Limit Theorem to estimate the probability $P(X < 110)$.

We assume that $(X - 120)/\sqrt{80}$ is approximately standard normal, so that

$$\begin{aligned} P(X < 110) &= P(X - 120 < -10) \\ &= P\left(\frac{X - 120}{\sqrt{80}} < -1.12\right) \\ &\approx \Phi(-1.12) = 1 - \Phi(1.12) = 1 - 0.8686 = \boxed{13.14\%.} \end{aligned}$$

(c) Use the Central Limit Theorem to estimate the probability $P(\overline{X} > 5.5)$.

We assume that $(\overline{X} - 6)/\sqrt{1/5}$ is approximately standard normal, so that

$$\begin{aligned} P(\overline{X} > 5.5) &= P(\overline{X} - 6 > -0.5) \\ &= P\left(\frac{\overline{X} - 6}{\sqrt{1/5}} > -1.12\right) \\ &\approx 1 - \Phi(-1.12) = \Phi(1.12) = 0.8686 = \boxed{86.86\%.} \end{aligned}$$

Problem 5. Suppose that a coin with $P(H) = p = 1/3$ is flipped $n = 6$ times. Let X be the number of heads that show up.

(a) Compute the exact value of $P(3 \leq X \leq 4)$.

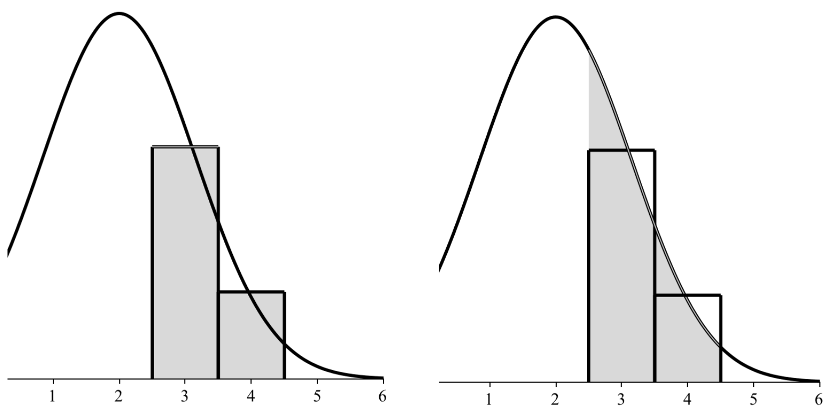
$$\begin{aligned} P(3 \leq X \leq 4) &= P(X = 3) + P(X = 4) \\ &= \binom{6}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + \binom{6}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 = \boxed{30.18\%.} \end{aligned}$$

(b) Compute an approximate value for $P(3 \leq X \leq 4)$ by integrating under a normal curve. Don't forget to use a continuity correction.

Let $X' \sim N(np, npq) = N(2, 4/3)$. Assuming that $X \approx X'$ gives

$$
\begin{aligned}
P(3 \le X \le 4) &\approx P(2.5 < X' < 4.5) \\
&= P(0.5 < X' - 2 < 2.5) \\
&= P\left(0.43 < \frac{X' - 2}{\sqrt{4/3}} < 2.17\right) \\
&= \Phi(2.17) - \Phi(0.43) = 0.9850 - 0.6664 = \boxed{31.86\%.}
\end{aligned}
$$

(c) Draw a picture comparing your answers from parts (a) and (b).



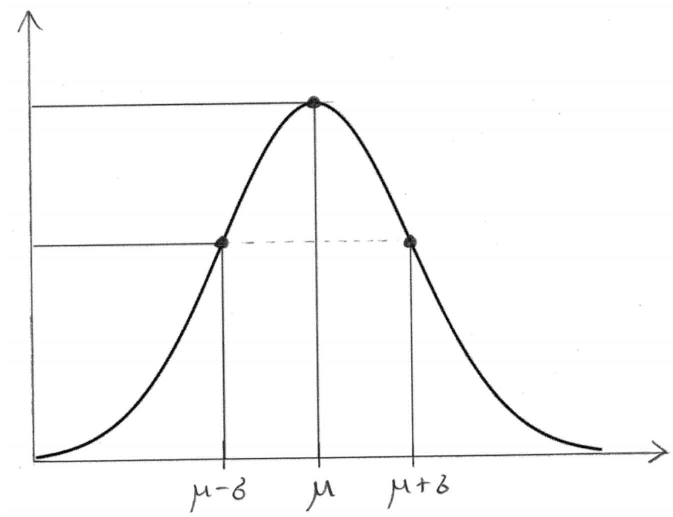
Version B

Problem 1. Let X be a **normal** random variable with mean μ and variance σ^2.

(a) Tell me the probability density function f_X.

$$
f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2}
$$

(b) Sketch the graph of f_X, showing the maximum and the points of inflection.



(c) Compute the probability $P(\mu - \sigma < X < \mu + 2\sigma)$.

Since $(X-\mu)/\sigma$ is standard normal, we have

$$\begin{aligned}P(\mu - \sigma < X < \mu + 2\sigma) &= P(-\sigma < X - \mu < 2\sigma) \\ &= P\left(-1 < \frac{X-\mu}{\sigma} < 2\right) \\ &= \Phi(2) - \Phi(-1) \\ &= \Phi(2) - [1 - \Phi(1)] \\ &= 0.9772 - [1 - 0.8413] = 81.85\%.\end{aligned}$$

Problem 2. Let X be a continuous random variable with the following pdf:

$$f_X(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $\mu = E[X]$ and $\sigma^2 = \mathrm{Var}(X)$.

We have

$$\mu = E[X] = \int_0^1 x \cdot \frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_0^1 = \boxed{3/8}$$

and

$$E[X^2] = \int_0^1 x^2 \cdot \frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{1}{5},$$

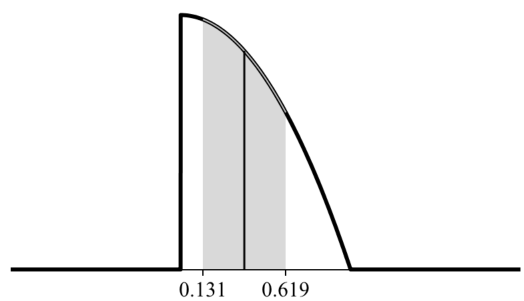
hence $\sigma^2 = \mathrm{Var}(X) = E[X^2] - \mu^2 = 1/5 - (3/8)^2 = \boxed{19/320.}$

(b) Compute the probability $P(\mu - \sigma < X < \mu + \sigma)$.

Since $\mu = 3/8$ and $\sigma = \sqrt{9/320} = 0.244$ we have

$$\begin{aligned} P(\mu - \sigma < X < \mu + \sigma) &= P(0.131 < X < 0.619) \\ &= \int_{0.131}^{0.619} \frac{3}{2}(1 - x^2)\, dx \\ &= \frac{3}{2}\left(x - \frac{x^3}{3}\right)\Bigg|_{0.131}^{0.619} = \boxed{61.37\%.} \end{aligned}$$

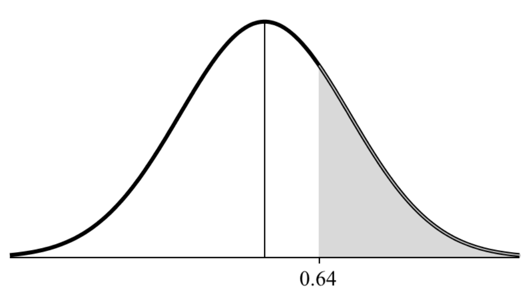
(c) Draw the graph of f_X, showing the region whose area you computed in part (b).



Problem 3. Let $Z \sim N(0, 1)$ be a standard normal random variable.

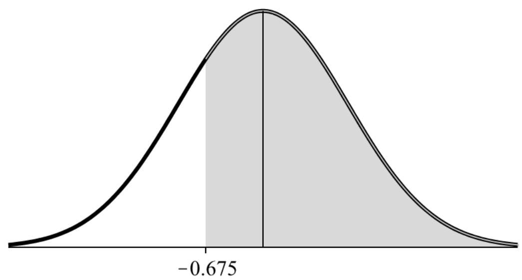
(a) Find α such that $P(Z > 0.64) = \alpha$ and draw a picture to illustrate your answer.

The area of the shaded region is $\alpha = 1 - \Phi(0.64) = 1 - 0.7389 = 0.2611$:



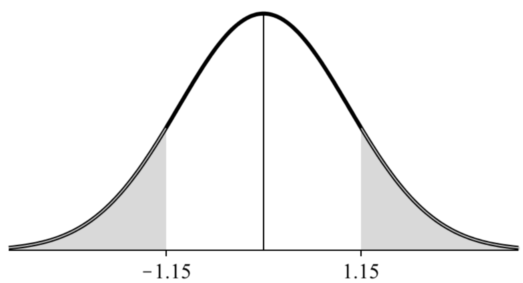
(b) Find c such that $P(Z > c) = 0.75$ and draw a picture to illustrate your answer.

We have $c = -0.675$. The area of the shaded region is 0.75:



(c) Find d such that $P(|Z| > d) = 0.25$ and draw a picture to illustrate your answer.

We have $d = 1.15$. The area of the shaded region is 0.25:



Problem 4. Let $X_1, X_2, \ldots, X_{20}$ be an iid sequence of random variables with

$$\mu = E[X_i] = 5 \qquad \text{and} \qquad \sigma^2 = \text{Var}(X_i) = 4.$$

Consider the sum X and the average $\overline{X}$ of these random variables:

$$X = X_1 + X_2 + \cdots + X_{20} \qquad \text{and} \qquad \overline{X} = \frac{1}{20} \cdot X.$$

(a) Compute the numbers $E[X]$, $\text{Var}(X)$, $E[\overline{X}]$ and $\text{Var}(\overline{X})$.

$$\begin{aligned} E[X] &= 20 \cdot \mu = \boxed{100,} \\ \text{Var}(X) &= 20 \cdot \sigma^2 = \boxed{80,} \\ E[\overline{X}] &= \mu = \boxed{5,} \\ \text{Var}(\overline{X}) &= \sigma^2/20 = 4/20 = \boxed{1/5.} \end{aligned}$$

(b) Use the Central Limit Theorem to estimate the probability $P(X > 110)$.

We assume that $(X - 100)/\sqrt{80}$ is approximately standard normal, so that

$$\begin{aligned} P(X > 110) &= P(X - 100 > 10) \\ &= P\left(\frac{X - 100}{\sqrt{80}} > 1.12\right) \\ &\approx 1 - \Phi(1.12) = 1 - 0.8686 = \boxed{13.14\%.} \end{aligned}$$

(c) Use the Central Limit Theorem to estimate the probability $P(\overline{X} < 4)$.

We assume that $(\overline{X} - 5)/\sqrt{1/5}$ is approximately standard normal, so that

$$\begin{aligned} P(\overline{X} < 4) &= P(\overline{X} - 5 < -1) \\ &= P\left(\frac{\overline{X} - 6}{\sqrt{1/5}} < -2.24\right) \\ &\approx \Phi(-2.24) = 1 - \Phi(2.24) = 1 - 0.9875 = \boxed{1.25\%.} \end{aligned}$$

Problem 5. Suppose that a coin with $P(H) = p = 2/3$ is flipped $n = 6$ times. Let X be the number of heads that show up.

(a) Compute the exact value of $P(3 \leq X \leq 4)$.

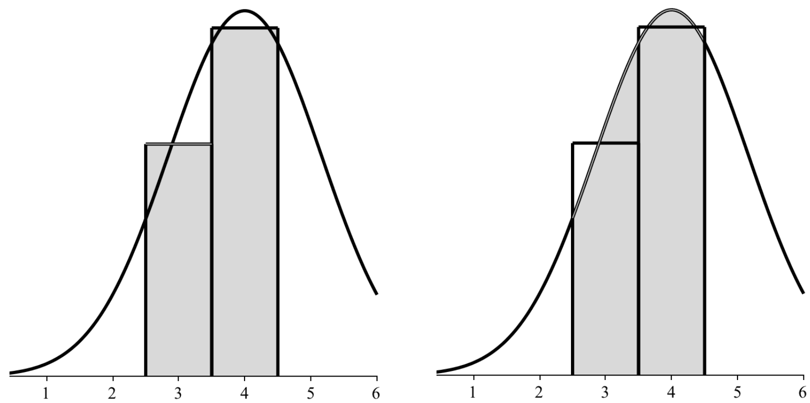
$$\begin{aligned} P(3 \leq X \leq 4) &= P(X = 3) + P(X = 4) \\ &= \binom{6}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 + \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 = \boxed{54.87\%.} \end{aligned}$$

(b) Compute an approximate value for $P(3 \leq X \leq 4)$ by integrating under a normal curve. Don't forget to use a continuity correction.

Let $X' \sim N(np, npq) = N(4, 4/3)$. Assuming that $X \approx X'$ gives

$$\begin{aligned} P(3 \leq X \leq 4) &\approx P(2.5 < X' < 4.5) \\ &= P(-1.5 < X' - 4 < 0.5) \\ &= P\left(-1.30 < \frac{X' - 2}{\sqrt{4/3}} < 0.43\right) \\ &= \Phi(0.43) - \Phi(-1.30) \\ &= \Phi(0.43) + [1 - \Phi(1.30)] \\ &= 0.6664 - [1 - 0.9032] = \boxed{56.96\%.} \end{aligned}$$

(c) Draw a picture comparing your answers from parts (a) and (b).


1 2 3 4 5 6
1 2 3 4 5 6