1. Let U be the uniform random variable on the interval $[2,5]$. Compute the following:

$$P(U<3), \quad P(3<U<6), \quad \mu=E[U], \quad \sigma^2=\mathrm{Var}(U), \quad P(\mu-\sigma<U<\mu+\sigma).$$

Recall that the density is

$$f_U(x)=\begin{cases}1/3 & 2\leq x\leq 5,\\ 0 & \text{otherwise.}\end{cases}$$

To compute each probability we must integrate the density. Since the density has a piecewise formula we must break up the integral over the appropriate intervals:

$$P(U<3)=\int_{-\infty}^{3} f_U(x)\,dx=\int_{-\infty}^{2} 0\,dx+\int_2^3 1/3\,dx=0+1/3=1/3.$$

$$P(3<U<6)=\int_3^6 f_U(x)\,dx=\int_3^5 1/3\,dx+\int_5^6 0\,dx=2/3+0=2/3.$$

Remark: The integral of a constant is $\int_a^b c\,dx=c[x]_a^b=c(b-a)$. We can also view this as the area of a rectangle with base $b-a$ and height c.

The first moment is

$$\begin{aligned}E[U]&=\int_{-\infty}^{\infty} x\cdot f_U(x)\,dx\\ &=\int_2^5 x(1/3)\,dx\\ &=(1/3)[x^2/2]_2^5\\ &=(1/3)[25/2-4/2]\\ &=7/2.\end{aligned}$$

which we could have guessed because the distribution is symmetric about 3.5. The second moment is

$$\begin{aligned}E[U^2]&=\int_{-\infty}^{\infty} x^2\cdot f_U(x)\,dx\\ &=\int_2^5 x^2(1/3)\,dx\\ &=(1/3)[x^3/3]_2^5\\ &=(1/3)[125/3-8/3]\\ &=13,\end{aligned}$$

and the variance is

$$\mathrm{Var}(U)=E[U^2]-E[U]^2=13-(7/2)^2=3/4.$$

Finally, we compute the probability that U falls between $\mu - \sigma$ and $\mu + \sigma$. Sine $\mu = 7/2$ and $\sigma = \sqrt{3/4}$ we have

$$\begin{aligned}
P(\mu - \sigma < U < \mu + \sigma) &= P(7/2 - \sqrt{3/4} < U < 7/2 + \sqrt{3/4}) \\
&= \int_{7/2-\sqrt{3/4}}^{7/2+\sqrt{3/4}} (1/3)\, dx \\
&= (1/3)[x]_{7/2-\sqrt{3/4}}^{7/2+\sqrt{3/4}} \\
&= (1/3)[(7/2 + \sqrt{3/4}) - (7/2 - \sqrt{3/4})] \\
&= (1/3)[2\sqrt{3/4}] \\
&\approx 57.7\%.
\end{aligned}$$

Remark: We would have gotten the same result for the uniform random variable on any interval. See the course notes for a proof.

2. Let X be a continuous random variable with the following density:

$$f_X(x) = \begin{cases} c \cdot \sin(x) & 0 \le x \le \pi, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.
(b) Compute $\mu = E[X]$ and $\sigma^2 = \text{Var}(X)$.[1]
(c) Compute $P(\mu - \sigma < X < \mu + \sigma)$.
(d) Draw a picture of the whole situation.

(a): The total mass is 1:

$$1 = \int_0^\pi c \cdot \sin(x)\, dx = c[-\cos(x)]_0^\pi = c[-\cos(\pi) + \cos(0)] = c[-(-1) + 1] = 2c.$$

Hence $c = 2$.

(b): If you don't remember integration by parts you can use the provided formulas. The first moment is

$$\begin{aligned}
\mu = E[X] &= \int_0^\pi x \cdot \sin(x)/2\, dx \\
&= (1/2)[\sin(x) - x\cos(x)]_0^\pi \\
&= (1/2)[\sin(\pi) - \pi\cos(\pi) - \sin(0) + 0\cos(1)] \\
&= (1/2)[0 + \pi - 0 + 0] \\
&= \pi/2.
\end{aligned}$$

[1]Hint: $\int x\sin(x)\, dx = \sin(x) - x\cos(x)$ and $\int x^2 \sin(x)\, dx = 2\cos(x) + 2x\sin(x) - x^2\cos(x)$.

We could have guessed this because the density is symmetric about $\pi/2$. Now the second moment and the variance:

$$\begin{aligned}
E[X^2] &= \int_0^{\pi} x^2 \cdot \sin(x)/2\, dx \\
&= (1/2)[2\cos(x) + 2x\sin(x) - x^2\cos(x)]_0^{\pi} \\
&= (1/2)[2\cos(\pi) + 2\pi\sin(\pi) - \pi^2\cos(\pi) - 2\cos(0) - 2\cdot 0\sin(0) - 0^2\cos(0)] \\
&= (1/2)[-2 + 0 + \pi^2 - 2 + 0 + 0] \\
&= (\pi^2 - 4)/2
\end{aligned}$$

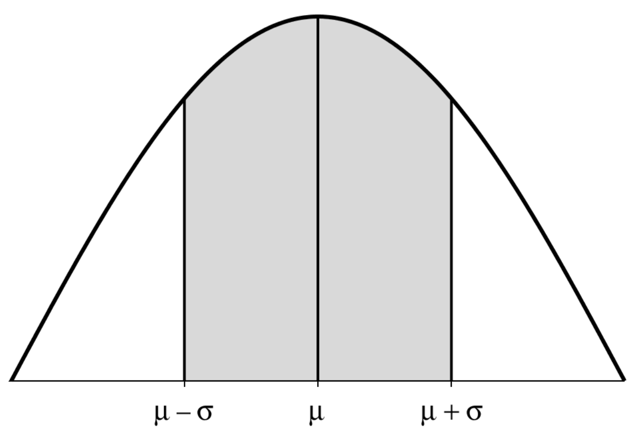
and

$$\sigma^2 = \mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{\pi^2 - 4}{2} - \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2 - 8}{4}.$$

(c): To compute $P(\mu - \sigma < X < \mu + \sigma)$ we should use a computer:

$$\begin{aligned}
P(\mu - \sigma < X < \mu + \sigma) &= \int_{\mu-\sigma}^{\mu+\sigma} \sin(x)/2\, dx \\
&= (1/2)[-\cos(\mu + \sigma) + \cos(\mu - \sigma)] \\
&\approx (1/2)[-\cos(2.254) + \cos(0.887)] \\
&\approx 63.2\%.
\end{aligned}$$

(d): Here is a picture:



3. Mean and Variance of a Normal Density. Let $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0, 1)$. In other words, suppose that X and Z have the following densities:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2} \quad \text{and} \quad f_Z(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}.$$

(a) Compute the expected value $E[Z]$. [Hint: Substitute $u = -z^2/2$.]
(b) Compute the second moment $E[Z^2]$ and the variance $\mathrm{Var}(Z)$. [Hint: Use integration by parts with $u = -z$ and $v = e^{-z^2/2}$. You may assume that $\int f_Z(z)\, dz = 1$.]
(c) Use parts (a) and (b) to compute $E[X]$ and $\mathrm{Var}(X)$. [Hint: We showed in class that $(X - \mu)/\sigma$ and Z have the same density.]

(a): The hint is fine, but maybe it's easier to use the fact that $z \cdot f_Z(z)$ is an **odd function**. Hence for any symmetric interval we have

$$\int_{-n}^{n} z \cdot f_Z(z)\,dz = 0.$$

Then taking $n \to \infty$ gives

$$E[Z] = \int_{-\infty}^{\infty} z \cdot f_Z(z)\,dz = \lim_{n\to\infty} \int_{-n}^{n} z \cdot f_Z(z)\,dz = \lim_{n\to\infty} 0 = 0.$$

(b): This one uses integration by parts. Let us take $u = -z$ and $v = e^{-z^2/2}$ so that

$$uv = -ze^{-z^2/2}, \quad udv = z^2e^{-z^2/2}dz, \quad \text{and} \quad vdu = -e^{-z^2/2}dz.$$

Then since $d(uv) = udv + vdu$ we have

$$\begin{aligned}\int udv + \int vdu &= uv \\ \int udv &= uv - \int vdu \\ \int z^2e^{-z^2/2}\,dz &= -ze^{-z^2/2} + \int e^{-z^2/2}\,dz.\end{aligned}$$

Now we divide everything by the constant $\sqrt{2\pi}$ and integrate from $-\infty$ to ∞:

$$\begin{aligned}\int_{-\infty}^{\infty} z^2 \cdot \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz &= \frac{1}{\sqrt{2\pi}} \cdot \left[-ze^{-z^2/2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz \\ E[Z^2] &= 0 + 1.\end{aligned}$$

This follows from the fact that $ze^{-z^2/2} \to 0$ as $z \to \pm\infty$ and $\int f_Z(z)\,dz = 1$.[2] Thus we obtain the variance:

$$\text{Var}(Z) = E[Z^2] - E[Z]^2 = 1 - 0^2 = 1.$$

(c): Now we consider the general normal density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2}.$$

Instead of performing bad integrals to compute $E[X]$ and $\text{Var}(X)$ we will use the fact (proved in the notes) that $(X-\mu)/\sigma$ and Z have the same density. It follows that

$$E[X] = E[\sigma Z + \mu] = \sigma E[Z] + \mu = 0\sigma + \mu = \mu$$

and

$$\text{Var}(X) = \text{Var}(\sigma Z + \mu) = \sigma^2\text{Var}(Z) = \sigma^2 \cdot 1 = \sigma^2.$$

Remark: Of course this must be the case, but the proof was surprisingly difficult. It turns out that normal random variables are easier to analyze with the technique of "moment generating functions". You will learn this technique in MTH 524.

4. Let $Z \sim N(0,1)$ so that $P(Z \le z) = \Phi(z)$. Use the attached table to compute the following probabilities:

---

[2] The limit has the indeterminate form $\infty \cdot 0$. I suppose this can be evaluated with L'Hopital's rule, but we can also use the heuristic that the exponential $e^{-z^2/2}$ goes to zero much faster than z goes to infinity, so the exponential will win.

(a) $P(Z < -0.5)$
(b) $P(0.33 < Z < 1.25)$
(c) $P(Z > 1)$, $P(Z > 2)$, $P(Z > 3)$
(d) $P(|Z| < 1)$, $P(|Z| < 2)$, $P(|Z| < 3)$

We repeatedly use the following facts:

$$P(z_1 < Z < z_2) = \Phi(z_2) - \Phi(z_1), \quad \Phi(z < Z) = 1 - \Phi(z), \quad P(-z) = 1 - \Phi(z).$$

(a): $P(Z < -0.5) = 1 - \Phi(0.5) = 1 - (0.6915) = 30.85\%$

(b): $P(0.33 < Z < 1.25) = \Phi(1.25) - \Phi(0.33) = (0.8944) - (0.6293) = \%26.51$

(c): For each we use the formula $P(Z > z) = 1 - \Phi(z)$:

$$\Phi(Z > 1) = 1 - \Phi(1) = 1 - (0.8413) = 15.85\%$$
$$\Phi(Z > 2) = 1 - \Phi(2) = 1 - (0.9772) = 2.28\%$$
$$\Phi(Z > 3) = 1 - \Phi(3) = 1 - (0.9987) = 0.13\%$$

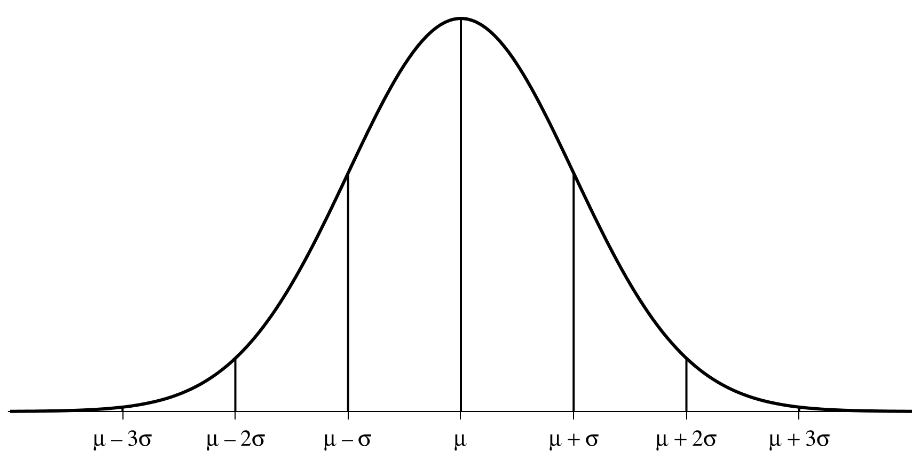
(d): First we note that

$$P(|Z| < z) = P(-z < Z < z) = \Phi(z) - \Phi(-z) = \Phi(z) - [1 - \Phi(z)] = 2\Phi(z) - 1.$$

Thus we have:

$$\Phi(|Z| < 1) = 2\Phi(1) - 1 = 2(0.8413) - 1 = 68.26\%$$
$$\Phi(|Z| < 2) = 2\Phi(2) - 1 = 2(0.9772) - 1 = 95.44\%$$
$$\Phi(|Z| < 3) = 2\Phi(3) - 1 = 2(0.9987) - 1 = 99.74\%$$

In summary, the probability that a normal random variable falls within 1, 2 or 3 standard deviations of its mean is approximately 68%, 95% and 99.7%, respectively. Here is a picture:



5. De Moivre-Laplace Consider a coin with $p = P(H) = 1/3$. Suppose that you flip the coin 100 times and let X be the number of times you get heads. Use the de Moive-Laplace theorem to compute the probability

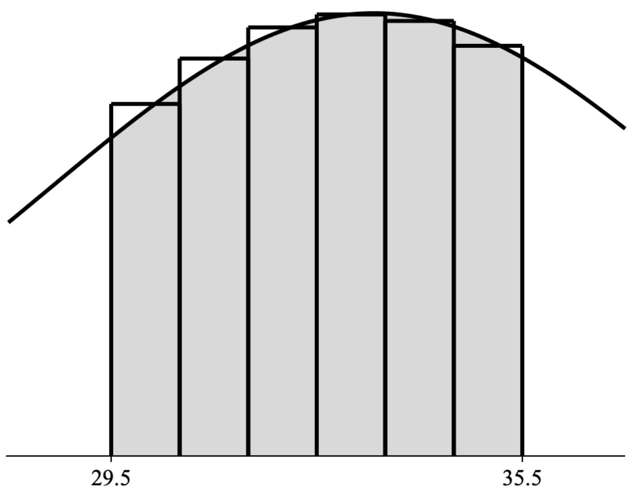
$$P(30 \leq X \leq 35).$$

Don't forget to use a continuity correction.

Since X has a binomial distribution we know that $E[X] = np = 100/3$ and $\text{Var}(X) = npq = 200/9$. Since neither of np or nq is close to zero it is reasonable to use the de Moivre-Laplace theorem, which says that $X \approx X'$ where $X' \sim N(100/3, 200/9)$ is a normal random variable. We use a continuity correction when transitioning from the discrete random variable X to the continuous random variable X'. Then we standardize X' to obtain a standard normal distribution and look up the result in our table:

$$
\begin{aligned}
P(30 \leq X \leq 35) &\approx P(29.5 < X' < 35.5) \\
&= P\left(\frac{29.5 - 100/3}{\sqrt{200/9}} < \frac{X' - 100/3}{\sqrt{2003}} < \frac{35.5 - 100/3}{\sqrt{200/9}}\right) \\
&\approx P(-0.81 < Z < 0.46) \\
&= \Phi(0.46) - \Phi(-0.81) \\
&= \Phi(0.81) - [1 - \Phi(0.46)] \\
&= (0.7910) - [1 - (0.6772)] \\
&= 46.82\%.
\end{aligned}
$$

Here is a picture. The area under the curve between 29.5 and 35.5 is an approximation for the area of the rectangles centered on 30 through 35:



6. Central Limit Theorem. Let $X_1, X_2, X_3, \ldots, X_{1000}$ be a sequence of iid random variables, each with mean $\mu = 600$ and variance $\sigma^2 = 40$. Consider the sample mean

$$\overline{X} = (X_1 + X_2 + \cdots + X_{1000})/1000.$$

Use the central limit theorem to estimate the probability that $\overline{X}$ falls between 599.9 and 600.1. [Remark: You should not use a continuity correction because the variables X_i are not necessarily discrete.]

If each sample X_i has mean $\mu = 600$ and variance $\sigma^2 = 40$ then the central limit theorem tells us that the sample mean $\overline{X}$ is approximately $N(\mu, \sigma^2/n)$, so[3]

$$\overline{X} \approx N(600, 40/1000).$$

[3]See the next problem for the proof that $E[\overline{X}] = \mu$ and $\text{Var}(\overline{X}) = \sigma^2/n$.

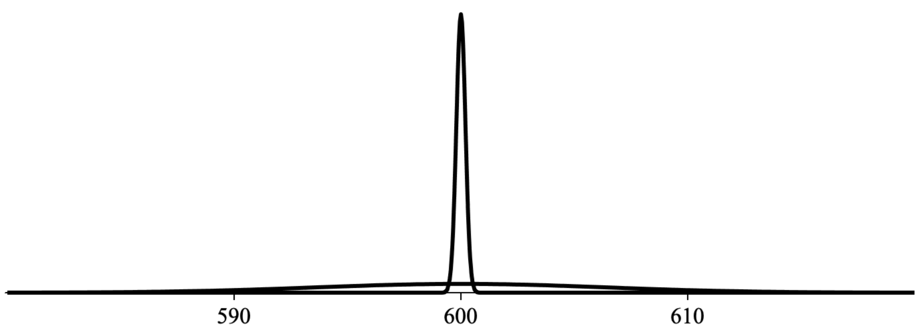
We standardize $\overline{X}$ to calculate the probability:

$$\begin{aligned} P(599.9 < \overline{X} < 600.1) &= P\left(\frac{599.9 - 600}{\sqrt{1/25}} < \frac{\overline{X} - 600}{\sqrt{1/25}} < \frac{600.1 - 600}{\sqrt{1/25}}\right) \\ &= P(-0.5 < Z < 0.5) \\ &= 2\Phi(0.5) - 1 \\ &= 38.30\%. \end{aligned}$$

To illustrate what this means, let us assume that each X_i is normal. Then each individual X_i has a vanishingly small probability of falling between 599.9 and 600.1:

$$\begin{aligned} P(599.9 < X_i < 600.1) &= P\left(\frac{599.9 - 600}{\sqrt{40}} < \frac{X_i - 600}{\sqrt{40}} < \frac{600.1 - 600}{\sqrt{40}}\right) \\ &= P(-0.02 < Z < 0.02) \\ &= 2\Phi(0.02) - 1 \\ &= 1.58\%. \end{aligned}$$

This is the effect of dividing the variance by 1000. Even though $\overline{X}$ and X_i have the same mean, $\overline{X}$ is much more concentrated around this mean. Here is a picture:



7. Sample Variance. Let $X_1, X_2, \ldots, X_n$ be a sequence of iid random variables, each with the same mean $\mu = E[X_i]$ and variance $\sigma^2 = \text{Var}(X_i)$. We define the *sample mean*

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the *sample variance*

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

(a) Show that $E[X_i^2] = \mu^2 + \sigma^2$.

(b) Show that $E[\overline{X}] = \mu$ and $\text{Var}(\overline{X}) = \sigma^2/n$ and use these to show that

$$E[\overline{X}^2] = \mu^2 + \sigma^2/n.$$

(c) Show that

$$\sum_{i=1}^{n}(X_i - \overline{X})^2 = \left(\sum_{i=1}^{n} X_i^2\right) - n\overline{X}^2.$$

[Hint: $\overline{X}\sum_i X_i = n\overline{X}^2$.]

(d) Use parts (a),(b),(c) and the linearity of expectation to show that $E[S^2] = \sigma^2$. This explains why we use $n-1$ instead of n in the denominator of the sample variance.

(a): We use the formula $\text{Var}(X_i) = E[X_i^2] - E[X_i]^2$ to get

$$E[X_i^2] = E[X_i]^2 + \text{Var}(X_i) = \mu^2 + \sigma^2.$$

(b): We use the linearity of expectation to get

$$\begin{aligned}
E[\overline{X}] &= E[(X_1 + X_2 + \cdots + X_n)/n] \\
&= E[(X_1 + X_2 + \cdots + X_n)]/n \\
&= (E[X_1] + E[X_2] + \cdots + E[X_n])/n \\
&= (\mu + \mu + \cdots + \mu)/n \\
&= (n\mu)/n \\
&= \mu.
\end{aligned}$$

We use the fact that the X_i are independent to get

$$\begin{aligned}
\text{Var}[\overline{X}] &= \text{Var}[(X_1 + X_2 + \cdots + X_n)/n] \\
&= \text{Var}[(X_1 + X_2 + \cdots + X_n)]/n^2 \\
&= (\text{Var}[X_1] + \text{Var}[X_2] + \cdots + \text{Var}[X_n])/n^2 \\
&= \left(\sigma^2 + \sigma^2 + \cdots + \sigma^2\right)/n^2 \\
&= (n\sigma^2)/n^2 \\
&= \sigma^2/n.
\end{aligned}$$

Hence we have

$$E[\overline{X}^2] = E[\overline{X}]^2 + \text{Var}(\overline{X}) = \mu^2 + \sigma^2/n.$$

(c): First we note that

$$\begin{aligned}
\left(\sum_{i=1}^{n} X_i\right)/n &= \overline{X} \\
\sum_{i=1}^{n} X_i &= n\overline{X},
\end{aligned}$$

so that

$$\overline{X}\sum_{i=1}^{n} X_i = \overline{X}n\overline{X} = n\overline{X}^2.$$

Then we have

$$\begin{aligned}\sum_{i=1}^{n}(X_i-\overline{X})^2 &= \sum_{i=1}^{n}(X_i^2-2\overline{X}X_i+\overline{X}^2)\\ &= \sum_{i=1}^{n}X_i^2-2\overline{X}\sum_{i=1}^{n}X_i+n\overline{X}^2\\ &= \sum_{i=1}^{n}X_i^2-2n\overline{X}^2+n\overline{X}^2\\ &= \sum_{i=1}^{n}X_i^2-n\overline{X}^2.\end{aligned}$$

(d): Finally, we have

$$\begin{aligned}E[S^2] &= E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2\right]\\ &= \frac{1}{n-1}\cdot E\left[\sum_{i=1}^{n}(X_i-\overline{X})^2\right]\\ &= \frac{1}{n-1}\cdot E\left[\sum_{i=1}^{n}X_i^2-n\overline{X}^2\right] && \text{from (c)}\\ &= \frac{1}{n-1}\cdot\left(\sum_{i=1}^{n}E[X_i^2]-n\cdot E[\overline{X}^2]\right)\\ &= \frac{1}{n-1}\cdot\left(\sum_{i=1}^{n}(\mu^2+\sigma^2)-n(\mu^2+\sigma^2/n)\right) && \text{from (a) and (b)}\\ &= \frac{1}{n-1}\cdot\big(n(\mu^2+\sigma^2)-n(\mu^2+\sigma^2/n)\big)\\ &= \frac{1}{n-1}\cdot\big(n\sigma^2-\sigma^2\big)\\ &= \frac{1}{n-1}\cdot(n-1)\sigma^2\\ &= \sigma^2.\end{aligned}$$

Remark: We have shown that the sample variance S^2 is an *unbiased estimator* for the population variance σ^2, which explains the $n-1$ in the denominator. It would me more obvious to consider the statistic

$$V=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2,$$

but this is a biased estimator for σ^2 because $V=\frac{n-1}{n}\cdot S^2$, and hence

$$E\,[V]=\frac{(n-1)^2}{n^2}\cdot E[S^2]=\frac{(n-1)^2}{n^2}\cdot\sigma^2\neq\sigma^2.$$