Math 224 | **Fall 2021**
Homework 2 | **Drew Armstrong**

---

1. A Florida license plate consists of six characters: **four letters followed by two numbers**. Characters are allowed to be repeated.

(a) Find the number of possible license plates.
(b) If a license plate is chosen at random, what is the probability that it contains at least one vowel $\{A, E, I, O, U\}$? [Hint: What if it contains **no** vowels?]

2. Suppose that a fair six-sided die has 3 sides painted red, 2 sides painted blue and 1 side painted green. Suppose you roll the die $n = 4$ times and let R, G, B be the number of times that you get red, green, blue, respectively.

(a) Compute $P(R = 1, G = 1, B = 2)$. [Hint: How many ways can it happen?]
(b) Compute $P(R \geq 1)$. [Hint: Think of the die as a coin.]
(c) Compute $P(G = B)$. [Hint: What are the possible values of R, G, B in this case?]

3. The Birthday Problem. Consider a classroom of r students. Each student has a birthday, which we can encode as a number from the set $\{1, 2, \ldots, 365\}$ (ignore leap years). Assume that each birthday is equally likely.

(a) Suppose that the r students are ordered (for example, alphabetically by last name). If we record each student's birthday, what is the size of the sample space?
(b) Compute the probability of the event $E =$"some pair of students have the same birthday". [Hint: Consider the opposite $E' =$"no two students have the same birthday".]
(c) Find the smallest number of students r such that $P(E) > 50\%$. [Use a computer.]

4. A Quick and Bad Proof of the Binomial Theorem.

(a) For all integers $r \geq 1$ show that $r! = r \times (r-1)!$. [Don't think too much.]
(b) For all integers $0 < k < n$, prove that

$$\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} = \frac{n!}{k!(n-k)!}.$$

[Hint: Use part (a) to get a common denominator.]

5. Suppose that 5 cards are dealt at random and without replacement from a standard deck of 52 cards.[1] Find the probabilities of the following events:

(a) 1 club, 1 diamond, 2 hearts, 1 spade
(b) 1 club, 1 spade, 3 red cards
(c) 2 black cards, 3 red cards

[Hint: The analysis is easier if you assume that the cards are not ordered, so the size of the sample space is "52 choose 5" = 2598960.]

---

[1] Each of the cards is labeled by one of four "suits" (clubs, diamonds, hearts, spades) and one of 13 "ranks" (1,2,. . . ,10,J,Q,K,A), for a total of $4 \times 13 = 52$ cards. Clubs and spades are "black cards"; diamonds and hearts are "red cards".

6. Two cards are drawn from a standard deck of 52 and placed sided by side on a table. Consider the following events:

$$A = \text{“the left card is a heart”},$$
$$B = \text{“the right card is black”}.$$

Compute the following probabilities:

$$P(A), \quad P(B), \quad P(B|A), \quad P(A \cap B), \quad P(A|B).$$

7. Bayes' Theorem. There are two bowls on a table. The first bowl contains 2 red chips and 3 green chips. The second bowl contains 4 red chips and 2 green chips. Your friend walks up to the table, chooses a bowl at random, and then chooses a chip at random. Assume that the two bowls are equally likely, and after having chosen a bowl, assume that the chips in that bowl are equally likely. Consider the events:

$$B_1 = \text{“the chip came from the first bowl”},$$
$$B_2 = \text{“the chip came from the second bowl”},$$
$$R = \text{“the chip is red”}.$$

(a) Compute the forwards probabilities $P(R|B_1)$ and $P(R|B_2)$.
(b) Compute the probability $P(R)$ that the chip is red. [Hint: The answer is not $6/11$ because the 11 chips in the two bowls are **not** equally likely.]
(c) Compute the backwards probability $P(B_1|R)$. That is, assuming that your friend chose a red chip, what is the probability that this chip came from the first bowl?