Math 224 **Fall 2021**

Homework 1 Solutions **Drew Armstrong**

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1. Suppose that a fair coin is flipped 5 times in sequence and let X be the number of "heads" that show up. Draw Pascal's triangle down to the sixth row (recall that the zeroth row consists of a single 1) and use your table to compute the probabilities $P(X = k)$ for $k = 0, 1, 2, 3, 4, 5$.

Here is Pascal's Triangle:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Then since $2^5 = 32$ we have the following table of probabilities:

k	0	1	2	3	4	5
$P(X = k)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

[Remark: If the coin were biased with $P(H) = p$ and $P(T) = q$ then probabilities would be

k	0	1	2	3	4	5
$P(X = k)$	q^5	$5pq^4$	$10p^2q^3$	$10p^3q^2$	$5p^4q$	p^5

Note that this gives the same answer when $p = q = 1/2$.]

2. Suppose that a fair coin is flipped 4 times in sequence.

(a) List all 16 outcomes in the sample space S.

(b) List the outcomes in each of the following events:

$A = \{\text{at least 1 head}\}$,

$B = \{\text{more than 2 heads}\}$,

$C = \{\text{heads on the 1st flip, anything on the other flips}\}$,

$D = \{\text{heads on the 1st and 2nd flips, anything on the other flips}\}$.

(c) Assuming that all outcomes are **equally likely**, use the formula $P(E) = \#E/\#S$ to compute the following probabilities:

$$P(A \cup B), \quad P(A \cap B), \quad P(C), \quad P(D), \quad P(C \cap D).$$

(a) The sample space is

$$\begin{aligned} S = \{&HHHH, \\ &HHHT, HHTH, HTHH, THHH, \\ &HHTT, HTHT, HTTH, THHT, THTH, TTHH, \\ &HTTT, THTT, TTHT, TTTH, \\ &TTTT\} \end{aligned}$$

(b) The events are

$$\begin{aligned} A = \{&HHHH,\\ &HHHT, HHTH, HTHH, THHH,\\ &HHTT, HTHT, HTTH, THHT, THTH, TTHH,\\ &HTTT, THTT, TTHT, TTTH\}, \end{aligned}$$

$$\begin{aligned} B = \{&HHHH,\\ &HHHT, HHTH, HTHH, THHH\}, \end{aligned}$$

$$\begin{aligned} C = \{&HHHH,\\ &HHHT, HHTH, HTHH,\\ &HHTT, HTHT, HTTH,\\ &HTTT\}, \end{aligned}$$

$$\begin{aligned} D = \{&HHHH,\\ &HHHT, HHTH,\\ &HHTT\}. \end{aligned}$$

(c) First observe that $B \subseteq A$ so that $A \cup B = A$ and $A \cap B = B$. Thus we have

$$P(A \cup B) = P(A) = \frac{\#A}{\#S} = \frac{15}{16} \quad \text{and} \quad P(A \cap B) = P(B) = \frac{\#B}{\#S} = \frac{5}{16}.$$

We also have

$$P(C) = \frac{\#C}{\#S} = \frac{8}{16} = \frac{1}{2} \quad \text{and} \quad P(D) = \frac{\#D}{\#S} = \frac{4}{16} = \frac{1}{4}.$$

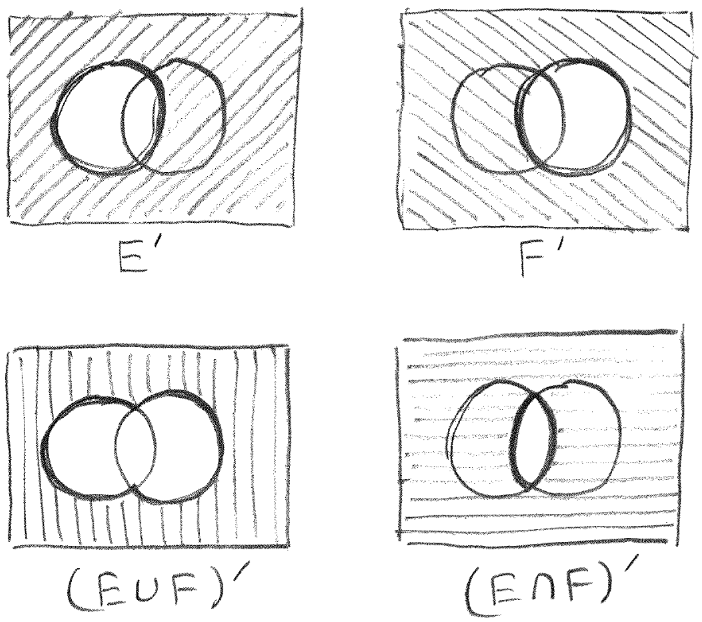
Finally, we note that $D \subseteq C$, so that $C \cap D = D$ and hence

$$P(C \cap D) = P(D) = \frac{1}{4}.$$

3. Draw Venn diagrams to verify *de Morgan's laws*: For all events $A, B \subseteq S$ we have

(a) $(E \cup F)' = E' \cap F'$,
(b) $(E \cap F)' = E' \cup F'$.

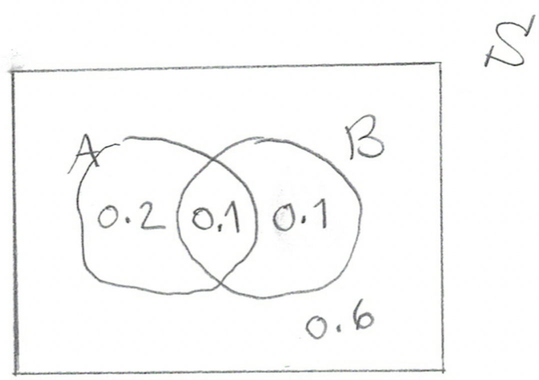
The proof follows by comparing the following four diagrams:



4. Let $A, B \subseteq S$ be two events satisfying $P(A) = 0.3$, $P(B) = 0.2$ and $P(A \cap B) = 0.1$. Use this information to compute the following probabilities. A Venn diagram may be helpful.

(a) $P(A \cup B)$,
(b) $P(A \cap B')$,
(c) $P(A \cup B')$.

It is easiest to draw a Venn diagram and use the given information to fill in the four regions:



This is like solving a puzzle. Then to obtain the probability of any event we add the probabilities of the corresponding regions:

$$P(A \cup B) = 0.2 + 0.1 + 0.1 = 0.4,$$
$$P(A \cap B') = 0.2,$$
$$P(A \cup B') = 0.2 + 0.1 + 0.6 = 0.9.$$

Alternatively, you can ignore the diagram and use pure algebra. First, the Principle of Inclusion-Exclusion gives

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.2 - 0.1 = 0.4.$$

Then the Law of Total Probability gives

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B') \\ 0.3 &= 0.1 + P(A \cap B') \\ 0.2 &= P(A \cap B'). \end{aligned}$$

Finally, we use Inclusion-Exclusion and Complementary Events:

$$\begin{aligned} P(A \cup B') &= P(A) + P(B') - P(A \cap B') \\ &= P(A) + [1 - P(B)] - P(A \cap B') \\ &= 0.3 + [1 - 0.2] - 0.2 \\ &= 0.9. \end{aligned}$$

These are just the steps that you did in your head when you filled in the diagram.

5. Suppose that you roll a pair of **fair** six-sided dice. For the sake of argument, let's suppose that one die is blue and the other is red, so we can tell the dice apart.

(a) Write down all elements of the sample space S. What is $\#S$?

(b) Compute the probability of getting a "double six", i.e., a six on each die. [Hint: Let $E \subseteq S$ be the set of outcomes that correspond to "double six". What is $\#E$? Assuming that all outcomes are equally likely, you can use the formula $P(E) = \#E/\#S$.]

(a) Let's suppose that one die is "blue" and the other is "red," so we can tell them apart. In other words, the outcome "12"="the blue die shows 1 and the red die shows 2" will differ from the outcome "21"="the blue die shows 2 and the red die shows 1." The the sample space is:

$$\begin{aligned} S = \{&11, 12, 13, 14, 15, 16 \\ &21, 22, 23, 24, 25, 26 \\ &31, 32, 33, 34, 35, 36 \\ &41, 42, 43, 44, 45, 46 \\ &61, 62, 63, 64, 65, 66\}. \end{aligned}$$

Independence and fairness suggest that for any outcome $ij \in S$ we must have $P(ij) = P(i)P(j) = (1/6)(1/6) = 1/36$. In other words, the 36 outcomes are equally likely.[1]

(b) Let $E =$"double six," so that $E = \{66\}$. Then we have

$$P(E) = \frac{\#E}{\#S} = \frac{1}{36}.$$

6. Consider a strange coin with $P(H) = 1/3$ and $P(T) = 2/3$. Suppose that you flip the coin 5 times and let X be the number of heads that you get. Find the probability $P(X \leq 4)$. [Hint: Observe that $P(X \leq 4) + P(X = 5) = 1$. Maybe it's easier to compute $P(X = 5)$.]

[1] It's perfectly okay to consider the two dice as "unordered" or "uncolored." Then we will have $\#S = 21$. However, in this case the outcomes will **not** be equally likely, which makes the analysis much harder.

There is only one way to get $X = 5$:

$$\text{“}X = 5\text{”} = \{HHHHH\}.$$

Since the coin flips are independent we obtain

$$\begin{aligned} P(X = 5) &= P(HHHHH) \\ &= P(H)P(H)P(H)P(H)P(H) \\ &= P(H)^5 \\ &= (1/3)^5 \\ &= 1/243 \text{ (or } 0.41\%) \end{aligned}$$

and hence $P(X \leq 4) = 1 - P(X = 5) = 242/243$ (or 99.59%). That's pretty likely.

7. Analyze the Chevalier de Méré's two experiments:

(a) Roll a **fair** six-sided die 4 times and let X be the number of "sixes" that you get. Compute $P(X \geq 1)$. [Hint: You can think of a die roll as a "strange coin flip", where $H =$"six" and $T =$"not six".]

(b) Roll a pair of **fair** six-sided dice 24 times and let Y be the number of "double sixes" that you get. Compute $P(Y \geq 1)$. [Hint: You can think of one roll of the dice as a "very strange coin flip", where $H =$"double six" and $T =$"not double six".]

[Hint: Problems 5 and 6 are relevant.]

(a) Roll a fair six-sided die and let $H =$"we get six," so that $P(H) = 1/6$ and $P(T) = 5/6$. Then following the logic of Problem 6 gives

$$P(X \geq 1) = 1 - P(X = 0) = 1 - P(T)^4 = 1 - \left(\frac{5}{6}\right)^4 = 51.77\%.$$

(b) Roll a pair of fair six-sided dice and let $H =$"we get double six." From Problem 5 we know that $P(H) = 1/36$ and $P(T) = 35/36$. Then following the logic of Problem 6 gives

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - P(T)^{24} = 1 - \left(\frac{35}{36}\right)^{24} = 49.14\%.$$

[Remark: This agrees with the Chevalier's experimental evidence that $P(X \geq 1)$ is slightly greater than 50% and $P(Y \geq 1)$ is slightly less than 50%.]