Math 224 **Exam 1**
Fall 2021 **Thurs Sept 23**

This is a closed book test. No electronic devices are allowed. There are 5 pages and 5 problems, each worth 6 points, for a total of 30 points.

Problem 1. Let S be the sample space of an experiment and consider two events $A, B \subseteq S$ with the following properties:

$$P(A) = 3/7, \qquad P(B) = 4/7 \qquad \text{and} \qquad P(A \cap B) = 1/7.$$

(a) Compute the probability $P(A \cup B)$ that A or B happens.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{7} + \frac{4}{7} - \frac{1}{7} = \frac{6}{7}.$$

(b) Compute the probability $P(A \cap B')$ that A happens but B does not happen.

$$P(A \cap B) + P(A \cap B') = P(A)$$
$$P(A \cap B') = P(A) - P(A \cap B) = \frac{3}{7} - \frac{1}{7} = \frac{2}{7}.$$

(c) Compute the probability $P(A|B)$ that A happens, assuming that B happens.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/7}{4/7} = \frac{1}{4}.$$

Problem 2. Consider a biased coin with $P(H) = p$ and $P(T) = q$. Suppose that the coin is flipped 5 times and let X be the number of heads.

(a) Compute the probability $P(X = 3)$ in terms of p and q..

$$P(X = 3) = \binom{5}{3} p^3 q^2 = \frac{5!}{3!2!} \cdot p^3 q^2 = 10 p^3 q^2$$

(b) Compute the probability $P(X \geq 1)$ in terms of p and q.

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{5}{0} p^0 q^5 = 1 - q^5$$

(c) Use your formula from part (b) to compute the probability of getting at least one "six" in five rolls of a **fair** six-sided die.

Let $H =$ "we get six" and $T =$ "we don't get six", so that $p = P(H) = 1/6$ and $q = P(T) = 5/6$. Then the formula from part (b) gives

$$P(X \geq 1) = 1 - q^5 = 1 - \left(\frac{5}{6}\right)^5 \quad \text{(or 59.8\%)}$$

Problem 3. An urn contains 2 red and 5 green balls. Suppose that 3 balls are drawn without replacement.

(a) What is the size of the sample space?

$$\#S = \binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 7 \cdot 5 = 35$$

(b) What is the probability of getting one red and two green balls?

Since the outcomes are equally likely, the probability is $\#E/\#S$ where $\#E$ is the number of ways to choose 1 red and 2 green balls:

$$P(\text{1 red and 2 green}) = \frac{\#E}{\#S} = \frac{\binom{2}{1}\binom{5}{2}}{\binom{7}{3}} = \frac{2 \cdot 10}{35} = \frac{4}{7}.$$

(c) What is the probability of getting no green balls?

If you get no green balls then you must get 3 red balls, which is impossible because there are only 2 red balls in the urn. If we write $\binom{2}{3} = 0$ then

$$P(\text{no green balls}) = \frac{\binom{2}{3}\binom{5}{0}}{\binom{7}{3}} = \frac{0\binom{5}{0}}{\binom{7}{3}} = 0.$$

Problem 4. A **fair** four-sided die has sides labeled a, b, c, d. Roll the die 3 times and let A, B, C, D be the number of times that sides a, b, c, d show up.

(a) What is the size of the sample space?

$$\#S = \underbrace{4}_{\text{1st roll}} \times \underbrace{4}_{\text{2nd roll}} \times \underbrace{4}_{\text{3rd roll}} = 4^3 = 64$$

(b) What is the number of words of length 3 that can be made by using each of the letters a, b, c exactly once? Use your answer to compute $P(A = 1, B = 1, C = 1, D = 0)$.

The number of words of length 3 using the letters a, b, c (d does not appear) is

$$\frac{3!}{1!1!1!0!} = 3! = 6,$$

hence probability of getting a, b, c in some order is

$$P(A = 1, B = 1, C = 1, D = 0) = \frac{6}{64} = \frac{3}{32}.$$

(c) What is the number of words of length 3 that can be made from two copies of "a" and one copy of "b"? Use your answer to compute $P(A = 2, B = 1, C = 0, D = 0)$.

The number of words of length 3 using the letters a, a, b (c and d do not appear) is

$$\frac{3!}{2!1!0!0!} = 3$$

hence probability of getting a, a, b in some order is

$$P(A = 2, B = 1, C = 0, D = 0) = \frac{3}{64}.$$

Remark: It is also possible to solve these problems with the multinomial formula. For all $i + j + k + \ell = 3$ we have

$$\begin{aligned} P(A = i, B = j, C = k, D = \ell) &= \frac{3!}{i!j!k!\ell!} \left(\frac{1}{4}\right)^i \left(\frac{1}{4}\right)^j \left(\frac{1}{4}\right)^k \left(\frac{1}{4}\right)^\ell \\ &= \frac{3!}{i!j!k!\ell!} \left(\frac{1}{4}\right)^{i+j+k+\ell} \\ &= \frac{3!}{i!j!k!\ell!} \left(\frac{1}{4}\right)^3 \\ &= \frac{3!}{i!j!k!\ell!} \Big/ 4^3. \end{aligned}$$

Problem 5. A diagnostic test is administered to a random person to determine if they have a certain disease. Consider the events $D =$ "the person has the disease" and $T =$"the test returns positive". Suppose that

$$P(T|D') = 1\%, \quad P(T'|D) = 2\% \quad \text{and} \quad P(D) = 30\%.$$

(a) Compute the probabilities $P(T'|D')$ and $P(T|D)$.

For all events A, B we have $P(A|B) + P(A'|B) = 1$. In our case:

$$\begin{aligned} P(T'|D') &= 1 - P(T|D') = 99\%, \\ P(T|D) &= 1 - P(T'|D) = 98\%. \end{aligned}$$

(b) Compute the probability $P(T)$. [Hint: Law of Total Probability.]

The law of total probability and the definition of conditional probability give

$$\begin{aligned} P(T) &= P(D \cap T) + P(D' \cap T) \\ &= P(D)P(T|D) + P(D')P(T|D') \\ &= (0.3)(.98) + (0.7)(0.01) \\ &= (3/10)(98/100) + (7/10)(1/100) \\ &= (294/1000) + (7/1000) \\ &= 301/1000 \\ &= 30.1\% \end{aligned}$$

(c) Compute the probability $P(D|T)$. [Hint: Bayes' Theorem.]

In part (b) we saw that $P(D \cap T) = 294/1000$ and $P(T) = 301/1000$. Then the definition of conditional probability gives

$$P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{294/1000}{301/1000} = \frac{294}{301} \quad \text{(or 97.7\%)}.$$

Remark: The hint was to use Bayes' Theorem but it wasn't really necessary because I guided you through all the steps. To use Bayes' Theorem explicitly, write

$$\begin{aligned} P(T)P(D|T) &= P(D)P(T|D) \\ P(D|T) &= P(D)P(T|D) \Big/ P(T), \end{aligned}$$

and then continue from there.