Math 224 **Fall 2017**
Homework 6 **Drew Armstrong**

Problems from 9th edition of *Probability and Statistical Inference* by Hogg, Tanis and Zimmerman:

- Section 7.1, Exercises 2, 4, 7
- Section 7.3, Exercises 1, 6, 8(a,b)

Additional Problems.

1. Sample Standard Deviation. Let $X_1, X_2, \ldots, X_n$ be independent samples from an underlying population with mean μ and variance σ^2. We have seen that the sample mean $\overline{X} = (X_1 + X_2 + \cdots + X_n)/n$ is an *unbiased estimator* for the population mean μ because

$$E[\overline{X}] = \mu.$$

The most obvious way to estimate the population variance σ^2 is to use the formula

$$V = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Unfortunately, you will show that this estimator is **biased**.

(a) Explain why $E[X_i^2] = \mu^2 + \sigma^2$ for each i.

(b) Use the linearity of expectation together with part (a) and the fact that $\sum X_i = n\overline{X}$ to show that

$$\begin{aligned} E[V] &= \frac{1}{n}\left(E[\sum X_i^2] - 2E[\overline{X}\sum X_i] + E[n\overline{X}^2]\right) \\ &= \frac{1}{n}\left(n(\mu^2+\sigma^2) - nE[\overline{X}^2]\right) \\ &= \mu^2 + \sigma^2 - E[\overline{X}^2] \end{aligned}$$

(c) Use the formula $\text{Var}(\overline{X}) = E[\overline{X}^2] - E[\overline{X}]^2$ to show that

$$E[\overline{X}^2] = \mu^2 + \sigma^2/n.$$

(d) Put everything together to show that

$$E[V] = \frac{n-1}{n}\cdot\sigma^2 \neq \sigma^2,$$

hence V is a **biased** estimator for σ^2.

It follows that the weird formula

$$S^2 = \frac{n}{n-1}\cdot V = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

satisfies

$$E[S^2] = E\left[\frac{n}{n-1}\cdot V\right] = \frac{n}{n-1}\cdot E[V] = \frac{\cancel{n}}{\cancel{n-1}}\cdot\frac{\cancel{n-1}}{\cancel{n}}\cdot\sigma^2 = \sigma^2$$

and hence S^2 is an **unbiased** estimator for σ^2. We call it the *sample variance* and we call its square root S the *sample standard deviation*. It is a sad fact that S is a **biased** estimator for σ but you will have to take more statistics courses if you want to learn about that.