Math 224 **Fall 2017**
Homework 5 **Drew Armstrong**

Problems from 9th edition of *Probability and Statistical Inference* by Hogg, Tanis and Zimmerman:

- Section 3.1, Exercises 3, 10.
- Section 3.3, Exercises 2, 3, 10, 11.
- Section 5.6, Exercises 2, 4.
- Section 5.7, Exerciese 4, 14.

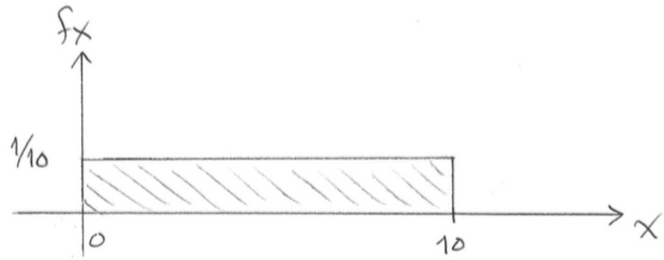
Solutions to Book Problems.

3.1-3. Customers arrive randomly at a bank teller's window. Given that a customer arrived in a certain 10-minute period, let X be the exact time within the 10 minutes that the customer arrived. We will assume that X is $U(0,10)$, i.e., that X is uniformly distributed on the real interval $[0,1] \subseteq \mathbb{R}$.

(a) Find the pdf of X. *Solution:*

$$f_X(x) = \begin{cases} 1/10 & \text{if } 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

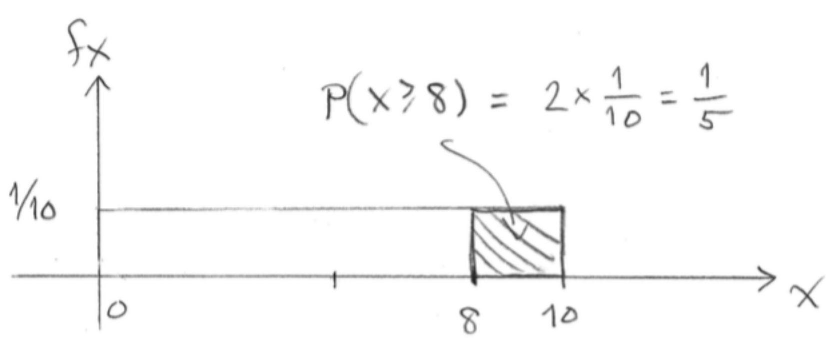
Here is a picture (not to scale):



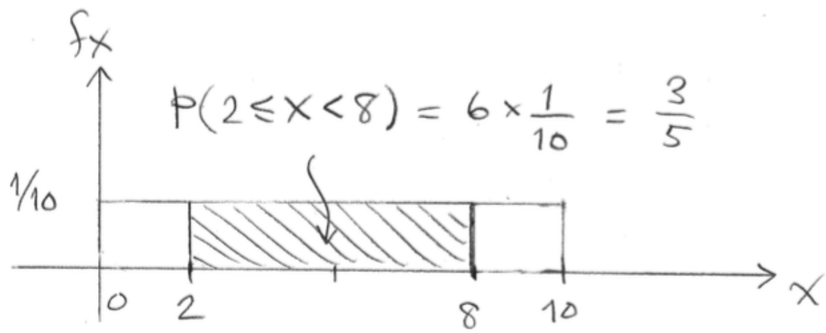
(b) Compute $P(X \geq 8)$. *Solution:* We could compute an integral:

$$P(X \geq 8) = \int_8^{10} 1/10\,dx = x/10\Big|_8^{10} = 10/10 - 8/10 = 2/10 = 1/5.$$

Or we could just recognize that this is the area of a rectangle with height 1/10 and width 2:



(c) Compute $P(2 \leq X < 8)$. *Solution:* Skipping the integral, we'll compute this as the area of a rectangle with width 2 and height 1/10:



Remark: For general $0 \leq a \leq b \leq 10$ we will have $P(a \leq X \leq b) = b - a$.

(d) Compute the expected value $E[X]$. *Solution:* We have

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{10} x/10 = x^2/20\Big|_0^{10} = 100/20 - 0/20 = 5.$$

Indeed, this agrees with our intuition that the distribution is symmetric about $x = 5$.

(e) Compute the variance $\mathrm{Var}(X)$. *Solution:* We could first compute $E[X^2]$ first, but instead we'll go directly from the definition. Since $\mu = 5$ we have

$$\begin{aligned}
\mathrm{Var}(X) = E[(X-\mu)^2] &= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx \\
&= \int_0^{10} (x-5)^2/10\, dx \\
&= \int_0^{10} (x^2 - 10x + 25)/10\, dx \\
&= (x^3/3 - 10x^2/2 + 25x)/10\Big|_0^{10} \\
&= (1000/3 - 1000/2 + 250)/10 \\
&= (2000/6 - 3000/6 + 1500/6)/10 \\
&= 50/6.
\end{aligned}$$

Indeed, if $X \sim U(a,b)$ then the front of the book says that $\mathrm{Var}(X) = (b-a)^2/12$, which agrees with our answer when $a = 0$ and $b = 10$.

3.1-10. The pdf[1] of X is $f(x) = c/x^2$ with support $1 < x < \infty$. (This means that the function is zero outside of this range.)

[1]The textbook is lying here because we don't know yet whether this really is a pdf.

(a) Calculate the value of c so that $f(x)$ is a pdf. *Solution:* We must have

$$\begin{aligned}
1 &= \int_1^\infty f(x)\,dx \\
&= \int_1^\infty c/x^2\,dx \\
&= -c/x\Big|_0^\infty \\
&= 0 - (-c/1) \\
&= c.
\end{aligned}$$

(b) Show that $E[X]$ is not finite. *Solution:* If the expected value existed then it would satisfy the formula

$$E[X] = \int_1^\infty x f(x)\,dx = \int_1^\infty 1/x\,dx.$$

But the antiderivative of $1/x$ is the natural logarithm $\log(x)$, so that

$$\int_1^\infty 1/x\,dx = \left[\lim_{x\to\infty} \log(x)\right] - \log(1) = \left[\lim_{x\to\infty} \log(x)\right] = \infty.$$

If the random variable X represents some kind of waiting time, then we should expect to wait forever!

[Moral of the Story: The expected value and variance are useful tools. However: (1) Some continuous random variables X have an infinite expected value $E[X] = \infty$. (2) Some random variables with finite expected value $E[X] < \infty$ still have infinite variance $\mathrm{Var}(X) = \infty$. So be careful.]

3.3-2. If $Z \sim N(0,1)$ has a standard normal distribution, compute the following probabilities. We will use the general formulas

- $P(a \leq Z \leq b) = \Phi(b) - \Phi(a)$
- $\Phi(-z) = 1 - \Phi(z)$

and we will look up the values for $\Phi(z)$ in the table on page 494 of the textbook.

(a)

$$P(0 \leq Z \leq 0.87) = \Phi(0.87) - \Phi(0) = 0.8078 - 0.5000 = 30.78\%$$

(b)

$$\begin{aligned}
P(-2.64 \leq Z \leq 0) &= \Phi(0) - \Phi(-2.64) \\
&= \Phi(0) - [1 - \Phi(2.64)] \\
&= \Phi(2.64) + \Phi(0) - 1 \\
&= 0.9959 + 0.5000 - 1 = 49.59\%.
\end{aligned}$$

(c)

$$\begin{aligned}
P(-2.13 \leq Z \leq -0.56) &= \Phi(-0.56) - \Phi(-2.13) \\
&= [1 - \Phi(0.56)] - [1 - \Phi(2.13)] \\
&= \Phi(2.13) - \Phi(0.56) \\
&= 0.9834 - 0.7123 = 27.11\%.
\end{aligned}$$

(d)

$$\begin{aligned} P(|Z| > 1.39) &= P(Z > 1.39) + P(Z < -1.39) \\ &= 1 - \Phi(1.39) + \Phi(-1.39) \\ &= 1 - \Phi(1.39) + [1 - \Phi(1.39)] \\ &= 2\,[1 - \Phi(1.39)] = 2\,[1 - 0.9177] = 16.46\%. \end{aligned}$$

(e)

$$P(Z < -1.62) = \Phi(-1.62) = 1 - \Phi(1.62) = 1 - 0.9474 = 5.26\%.$$

(f)

$$\begin{aligned} P(|Z| > 1) &= P(Z > 1) + P(Z < -1) \\ &= 1 - \Phi(1) + \Phi(-1) \\ &= 1 - \Phi(1) + [1 - \Phi(1)] \\ &= 2\,[1 - \Phi(1)] = 2\,[1 - 0.8413] = 31.74\%. \end{aligned}$$

(g) After parts (d) and (f) we observe the general pattern:

$$\boxed{P(|Z| > z) = 2\,[1 - \Phi(z)]\,.}$$

Therefore we have

$$P(|Z| > 2) = 2\,[1 - \Phi(2)] = 2\,[1 - 0.9772] = 4.56\%$$

(h) and also

$$P(|Z| > 3) = 2\,[1 - \Phi(3)] = 2\,[1 - 0.9987] = 2.6\%.$$

3.3-3. Suppose $Z \sim N(0,1)$. Find values of c to satisfy the following equations.

(a) $P(Z \geq c) = 0.025$. *Solution:* We are looking for c such that

$$\begin{aligned} P(Z \geq c) &= 0.025 \\ 1 - P(Z \leq c) &= 0.025 \\ 1 - \Phi(c) &= 0.025 \\ 0.9750 &= \Phi(c). \end{aligned}$$

My trusty table tells me that $\Phi(1.96) = 0.975$, and hence $c = 1.96$.

(b) $P(|Z| \leq c) = 0.95$. *Solution:* We are looking for c such that

$$\begin{aligned} P(|Z| \leq c) &= 0.95 \\ P(-c \leq Z \leq c) &= 0.95 \\ \Phi(c) - \Phi(-c) &= 0.95 \\ \Phi(c) - [1 - \Phi(c)] &= 0.95 \\ 2\Phi(c) - 1 &= 0.95 \\ \Phi(c) &= 1.95/2 = 0.9750. \end{aligned}$$

So the answer is the same as for part (a), i.e., $c = 1.96$.

(c) $P(Z > c) = 0.05$. *Solution:* Following the same steps as in part (a) gives

$$\begin{aligned} P(Z > c) &= 0.05 \\ 1 - P(Z > c) &= 0.05 \\ 1 - \Phi(c) &= 0.05 \\ 0.9500 &= \Phi(c). \end{aligned}$$

We look up in the table that $\Phi(1.64) = 0.9495$ and $\Phi(1.65) = 0.9505$. Therefore we must have $\Phi(1.645) \approx 0.9500$ and hence $c \approx 1.645$.

(d) $P(|Z| \leq c) = 0.90$. *Solution:* Following the same steps as in part (b) gives

$$\begin{aligned} P(|Z| \leq c) &= 0.90 \\ P(-c \leq Z \leq c) &= 0.90 \\ \Phi(c) - \Phi(-c) &= 0.90 \\ \Phi(c) - [1 - \Phi(c)] &= 0.90 \\ 2\Phi(c) - 1 &= 0.90 \\ \Phi(c) &= 1.90/2 = 0.9500. \end{aligned}$$

So the answer is the same as for part (c), i.e., $c \approx 1.645$.

3.3-10. Let $X \sim N(\mu, \sigma^2)$ be normal and for any real numbers $a, b \in \mathbb{R}$ with $a \neq 0$ define the random variable

$$Y = aX + b.$$

By properties of expectation and variance we have

$$E[Y] = E[aX + b] = aE[X] + b = a\mu + b$$

and

$$\text{Var}(Y) = \text{Var}(aX + b) = \text{Var}(aX) = a^2\text{Var}(X) = a^2\sigma^2.$$

I claim, furthermore thatn Y is also **normal**, i.e., that

$$Y \sim N(a\mu + b, a^2\sigma^2).$$

Proof: To show that Y is normal, we want to show for any real numbers $y_1 \leq y_2$ that

$$(?) \qquad P(y_1 \leq Y \leq y_2) = \int_{w=y_1}^{w=y_2} \frac{1}{\sqrt{2\pi a^2\sigma^2}}\, e^{-(w-a\mu-b)^2/2a^2\sigma^2}\, dw.$$

To show this, we can use the fact that X is normal to obtain[2]

$$\begin{aligned} P(y_1 \leq Y \leq y_2) &= P(y_1 \leq aX + b \leq y_2) \\ &= P(y_1 - b \leq aX \leq y_2 - b) \\ &= P\left(\frac{y_1 - b}{a} \leq X \leq \frac{y_2 - b}{a}\right) \\ (*) \qquad &= \int_{x=(y_1-b)/a}^{x=(y_1-b)/a} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}\, dx. \end{aligned}$$

[2]In the third line here we will assume that $a > 0$. The proof for $a < 0$ is exactly the same except that it will switch the limits of integration.

To show that the expressions $(*)$ and $(?)$ are equal we will make the substitution

$$w = ax + b,$$
$$dw = a \cdot dx.$$

Then we observe that

$$\begin{aligned}\int_{w=y_1}^{w=y_2} \frac{1}{\sqrt{2\pi a^2\sigma^2}}\, e^{-(w-a\mu-b)^2/2a^2\sigma^2}\, dw &= \int_{x=(y_1-b)/a}^{x=(y_1-b)/a} \frac{1}{\sqrt{2\pi a^2\sigma^2}}\, e^{-(ax+\not b-a\mu-\not b)^2/2a^2\sigma^2}\, a\cdot dx \\ &= \int_{x=(y_1-b)/a}^{x=(y_1-b)/a} \frac{\not a}{\sqrt{2\pi \not a^{\not 2}\sigma^2}}\, e^{-\not a^{\not 2}(x-\mu)^2/2\not a^{\not 2}\sigma^2}\, dx \\ &= \int_{x=(y_1-b)/a}^{x=(y_1-b)/a} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}\, dx\end{aligned}$$

as desired. ///

[Remark: Sadly this proof is not very informative. We went to the trouble because we are very interested in the special case when $a = 1/\sigma$ and $b = -\mu/\sigma$. In this case the result becomes

$$X \sim N(\mu, \sigma^2) \qquad \Longrightarrow \qquad Y = \frac{X-\mu}{\sigma} \sim N(0,1).$$

We will use this fact in almost every problem below.]

3.3-11. A candy maker produces mints that have a label weight of 20.4 grams. We assume that the distribution of the weights of these mints is $N(21.37, 0.16)$.

(a) Let X denote the weight of a single mint selected at random from the production line. Find $P(X > 22.07)$.

Solution: Since $X \sim N(21.37, 0.16)$ we have $\mu = 21.37$ and $\sigma^2 = 0.16$, hence $\sigma = 0.4$. It follows from the remark just above that $(X - 21.37)/0.4$ has a standard normal distribution and hence

$$\begin{aligned}P(X > 22.07) &= P(X - 21.37 > 0.7) \\ &= P\left(\frac{X-21.37}{0.4} > 1.75\right) \\ &= 1 - P\left(\frac{X-21.37}{0.4} \leq 1.75\right) \\ &= 1 - \Phi(1.75) = 1 - 0.9599 = 4.01\%.\end{aligned}$$

(b) Suppose that 15 mints are selected independently and weighed. Let Y be the number of these mints that weigh less than 20.857 grams. Find $P(Y \leq 2)$.

Solution: Let $X_1, X_2, \ldots, X_{15}$ be the weights of the 15 randomly selected mints. By assumption each of these weights has distribution $N(21.37, 0.16)$ so that each random variable $(X_i - 21.37)/0.4$ is standard normal. For each i we have

$$\begin{aligned}P(X_i < 20.875) &= P(X_i - 21.37 < -0.531) \\ &= P\left(\frac{X_i - 21.37}{0.4} < -1.2825\right) \\ &\approx \Phi(-1.28) = 1 - \Phi(1.28) = 1 - 0.8995 = 10.05\%.\end{aligned}$$

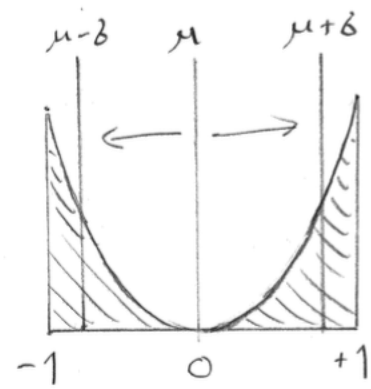
In other words, we can think of each of the 15 selected mints as a coin flip where "heads" means "the weight is less than 20.857" and the probability of heads is approximately 10%. Then Y is a binomial random variable with parameters $n = 15$ and $p \approx 0.1$ and we conclude that

$$\begin{aligned} P(Y \le 2) &\approx \sum_{k=0}^{2} \binom{15}{k} (0.1)^k (0.9)^{15-k} \\ &= (0.9)^{15} + 15 \cdot (0.1)(0.9)^{14} + 105 \cdot (0.1)^2 (0.9)^{13} \approx 81.59\%. \end{aligned}$$

In other words, there is an 80% chance that no more than 2 out of every 15 mints will weigh less than 20.857 grams. I don't know if that's good.

5.6-2. Let $Y = X_1 + X_2 + \cdots + X_{15}$ be the sum of a random sample of size 15 from a distribution whose pdf is $f(x) = (3/2)x^2$ with support $-1 < x < 1$. Using this pdf, one can use a computer to show that $P(-0.3 \le Y \le 1.5) = 0.22788$. On the other hand, we can use the Central Limit Theorem to approximate this probability.

Solution: The distribution in question looks like this:



Since the distribution is symmetric about zero, we conclude without doing any work that $\mu = E[X_i] = 0$ for each i. To find σ, however, we need to compute an integral. For any i, the variance of X_i is defined by

$$\begin{aligned} \sigma^2 = \mathrm{Var}(X_i) &= E\left[(X_i - 0)^2\right] \\ &= E[X_i^2] \\ &= \int_{-1}^{1} x^2 \cdot f(x)\, dx \\ &= \int_{-1}^{1} x^2 \cdot \frac{3}{2} x^2\, dx \\ &= \frac{3}{2} \int_{-1}^{1} x^4\, dx \\ &= \frac{3}{2} \cdot \frac{x^5}{5}\bigg|_{-1}^{1} = \frac{3}{2} \cdot \frac{1}{5} - \frac{3}{2} \cdot \frac{(-1)^5}{5} = \frac{6}{10} = \frac{3}{5}. \end{aligned}$$

It follows that Y has mean and variance given by

$$\mu_Y E[Y] = E[X_1] + E[X_2] + \cdots + E[X_{15}] = 0 + 0 + \cdots + 0 = 0$$

and

$$\sigma_Y^2 = \text{Var}(Y) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_{15}) = \frac{3}{5} + \frac{3}{5} + \cdots + \frac{3}{5} = 15 \cdot \frac{3}{5} = 9,$$

By the Central Limit Theorem, the sum Y is approximately normal and hence $(Y - \mu_Y)/\sigma_Y = Y/3$ is approximately standard normal. We conclude that

$$\begin{aligned}
P(-0.3 \leq Y \leq 0.5) &= P\left(\frac{-0.3}{3} \leq \frac{Y}{3} \leq \frac{1.5}{3}\right) \\
&= P\left(-0.1 \leq \frac{Y}{3} \leq 0.5\right) \\
&\approx \Phi(0.5) - \Phi(-0.1) \\
&= \Phi(0.5) - [1 - \Phi(0.1)] \\
&= \Phi(0.5) + \Phi(0.1) - 1 \\
&= 0.6915 + 0.5398 - 1 = 23.13\%.
\end{aligned}$$

That's reasonably close to the exact value 22.788%, I guess. We would get a more accurate result by taking more than 15 samples.

5.6-4. Approximate $P(39.75 \leq \overline{X} \leq 41.25)$, where $\overline{X}$ is the mean of a random sample of size 32 from a distribution with mean $\mu = 40$ and $\sigma^2 = 8$.

Solution: In general the sample mean is defined by $\overline{X} = (X_1 + X_2 + \cdots + X_n)/n$, where each X_i has $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$. From this we compute that

$$E[\overline{X}] = \frac{1}{n}\left(E[X_1] + E[X_2] + \cdots + E[X_n]\right) = \frac{1}{n}(\mu + \mu + \cdots + \mu) = \frac{n\mu}{n} = \mu$$

and

$$\text{Var}(\overline{X}) = \frac{1}{n^2}\left(\text{Var}(X_1) + \cdots + \text{Var}(X_n)\right) = \frac{1}{n^2}(\sigma^2 + \cdots + \sigma^2) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

The Central Limit Theorem says that if n is large then $\overline{X}$ is approximately normal:

$$\overline{X} \approx N(\mu, \sigma^2/n).$$

In the case $n = 32$, $\mu = 40$ and $\sigma^2 = 8$ we obtain

$$\overline{X} \approx N(40, 8/32) = N(40, (1/2)^2).$$

It follows that $(\overline{X} - 40)/(1/2) = 2(\overline{X} - 40)$ is approximately $N(0, 1)$ and hence

$$\begin{aligned}
P(39.75 \leq \overline{X} \leq 41.25) &= P\left(-0.25 \leq \overline{X} - 40 \leq 1.25\right) \\
&= P\left(-0.5 \leq 2(\overline{X} - 40) \leq 2.5\right) \\
&\approx \Phi(2.5) - \Phi(-0.5) \\
&= \Phi(2.5) - [1 - \Phi(0.5)] \\
&= \Phi(2.5) + \Phi(0.5) - 1 \\
&= 0.9938 + 0.6915 - 1 = 68.53\%.
\end{aligned}$$

5.7-4. Let X equal the number out of $n = 48$ mature aster seeds that will germinate when $p = 0.75$ is the probability that a particular seed germinates. Approximate $P(35 \leq X \leq 40)$.

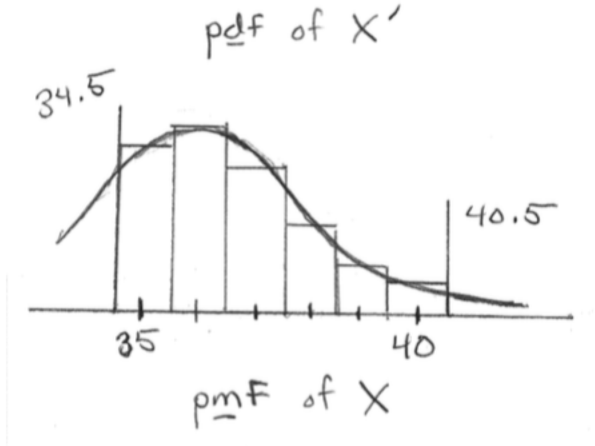
Solution: We observe that X is a binomial random variable with pmf

$$P(X = k) = \binom{48}{k}(0.75)^k(0.25)^{48-k}.$$

My laptop tells me that the exact probability is

$$P(35 \le X \le 40) = \sum_{k=35}^{40} P(X=k) = \sum_{k=35}^{40} \binom{48}{k} (0.75)^k (0.25)^{48-k} = 63.74\%.$$

If we want to compute an approximation by hand then we should use the de Moivre-Laplace Theorem (a special case of the Central Limit Theorem), which says that X is approximately normal with mean $np = 36$ and variance $\sigma^2 = np(1-p) = 9$, i.e., standard deviation $\sigma = 3$. Let X' be a **continuous** random variable with $X' \sim N(36, 3^2)$. Here is a picture comparing the probability **mass** function of the discrete variable X to the probability **density** function of the continuous variable X':



The picture suggests that we should use the following continuity correction:[3]

$$P(35 \le X \le 40) \approx P(34.5 \le X' \le 40.5).$$

And then because $(X'-36)/3$ is **standard** normal we obtain

$$\begin{aligned} P(34.5 \le X' \le 40.5) &= P(-1.5 \le X' - 36 \le 4.5) \\ &= P\left(-0.5 \le \frac{X'-36}{3} \le 1.5\right) \\ &= \Phi(1.5) - \Phi(-0.5) \\ &= \Phi(1.5) - [1 - \Phi(0.5)] \\ &= \Phi(1.5) + \Phi(0.5) - 1 = 0.9332 + 0.6915 - 1 = 62.47\% \end{aligned}$$

Not too bad.

5.7-14. A (fair six-sided) die is rolled 24 independent times. Let X_i be the number that appears on the ith roll and let $Y = X_1 + X_2 + \cdots + X_{24}$ be the sum of these numbers. The pmf of each X_i is given by the following table

k	1	2	3	4	5	6
$P(X_i = k)$	1/6	1/6	1/6	1/6	1/6	1/6

[3] If you don't do this then you will still get a reasonable answer, it just won't be as accurate.

So we find that:

$$E[X_i] = (1+2+3+4+5+6)/6 = 7/2,$$
$$E[X_i^2] = (1^2+2^2+3^2+4^2+5^2+6^2)/6 = 91/6,$$
$$\mathrm{Var}(X_i) = E[X_i^2] - E[X_i]^2 = 91/6 - (7/2)^2 = 35/12.$$

Then the expected value and variance of Y are given by

$$E[Y] = 24 \cdot E[X_i] = 24 \cdot \frac{7}{2} = 84 \quad \text{and} \quad \mathrm{Var}(Y) = 24 \cdot \mathrm{Var}(X_i) = 24 \cdot \frac{35}{12} = 70$$

and the Central Limit Theorem tells us that Y is approximately $N(84, 70)$. Let Y' be a continuous random variable that is exactly $N(84, 70)$, so that $(Y' - 84)/\sqrt{70}$ has a standard normal distribution.

(a) Compute $P(Y \geq 86)$. *Solution:*

$$\begin{aligned} P(Y \geq 86) &\approx P(Y' \geq 85.5) \\ &= P(Y' - 84 \geq 1.5) \\ &\approx P\left(\frac{Y' - 84}{\sqrt{70}} \geq 0.18\right) \\ &= 1 - \Phi(0.18) = 1 - 0.5714 = 42.86\%. \end{aligned}$$

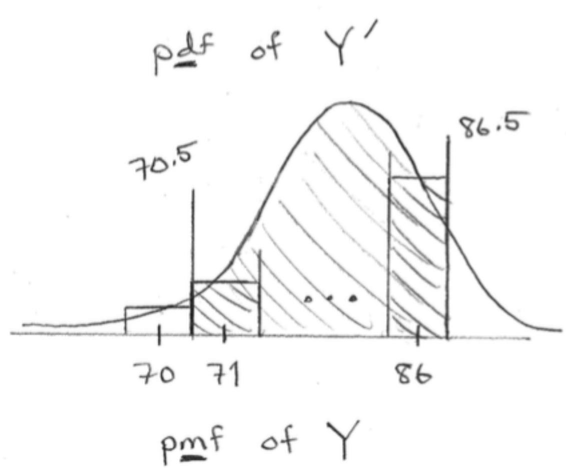
(b) Compute $(P < 86)$. *Solution:* This is the complement of part (a):

$$P(Y < 86) = 1 - P(Y \geq 86) \approx 1 - 0.4286 = 57.14\%.$$

(c) Compute $P(70 < Y \leq 86)$. *Solution:*

$$\begin{aligned} P(70 < Y \leq 86) &\approx P(70.5 \leq Y' \leq 86.5) \\ &= P(-13.5 \leq Y' - 84 \leq 2.5) \\ &\approx P\left(-1.61 \leq \frac{Y' - 84}{\sqrt{70}} \leq 0.30\right) \\ &= \Phi(0.30) - \Phi(-1.61) \\ &= \Phi(0.30) - [1 - \Phi(1.61)] \\ &= \Phi(0.30) + \Phi(1.61) - 1 = 0.6179 + 0.9463 - 1 = 56.42\%. \end{aligned}$$

Here is a picture explaining the continuity correction that we used in the first step:



Additional Problems.

1. The Normal Curve. Let $\mu, \sigma^2 \in \mathbb{R}$ be any real numbers (with $\sigma^2 > 0$) and consider the graph of the function

$$n(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}.$$

(a) Compute the first derivative $n'(x)$ and show that $n'(x) = 0$ implies $x = \mu$.
(b) Compute the second derivative $n''(x)$ and show that $n''(\mu) < 0$, hence the curve has a local maximum at $x = \mu$.
(c) Show that $n''(x) = 0$ implies $x = \mu + \sigma$ or $x = \mu - \sigma$, hence the curve has inflections at these points. [The existence of inflections at $\mu + \sigma$ and $\mu - \sigma$ was de Moivre's original motivation for defining the standard deviation.]

Solution: The chain rule tells us that for any function $f(x)$ we have

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot \frac{d}{dx} f(x).$$

Applying this to the function $n(x)$ gives

$$n'(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2} \cdot \frac{-1}{2\sigma^2} 2(x-\mu) = \boxed{\frac{-1}{\sqrt{8\pi\sigma^6}} \cdot e^{-(x-\mu)^2/2\sigma^2}} \cdot (x-\mu).$$

We observe that the expression inside the box is never zero. In fact, it is always strictly negative. Therefore we have $n'(x) = 0$ precisely when $(x - \mu) = 0$, or, in other words, when $x = \mu$. This tells us that there is a horizontal tangent when $x = \mu$. To determine whether this is a maximum or a minimum we should compute the second derivative. Using the product rule gives

$$\begin{aligned}
n''(x) &= \frac{d}{dx}\left[\frac{-1}{\sqrt{8\pi\sigma^6}} \cdot e^{-(x-\mu)^2/2\sigma^2} \cdot (x-\mu)\right] \\
&= \frac{-1}{\sqrt{8\pi\sigma^6}} \cdot \frac{d}{dx}\left[e^{-(x-\mu)^2/2\sigma^2} \cdot (x-\mu)\right] \\
&= \frac{-1}{\sqrt{8\pi\sigma^6}} \cdot \left[e^{-(x-\mu)^2/2\sigma^2} \cdot \frac{-1}{2\sigma^2} 2(x-\mu) \cdot (x-\mu) + e^{-(x-\mu)^2/2\sigma^2}\right] \\
&= \frac{-1}{\sqrt{8\pi\sigma^6}} \cdot e^{-(x-\mu)^2/2\sigma^2} \cdot \left[\frac{-(x-\mu)^2}{\sigma^2} + 1\right].
\end{aligned}$$

Then plugging in $x = \mu$ gives

$$n''(\mu) = \frac{-1}{\sqrt{8\pi\sigma^6}} < 0,$$

which implies that the graph of $n(x)$ curves down at $x = \mu$, so it must be a local **maximum**. Finally, we observe that the boxed formula in the following expression is always nonzero (in fact it is always negative):

$$n''(x) = \boxed{\frac{-1}{\sqrt{8\pi\sigma^6}} \cdot e^{-(x-\mu)^2/2\sigma^2}} \cdot \left[\frac{-(x-\mu)^2}{\sigma^2} + 1\right]$$

Therefore we have $n''(x) = 0$ precisely when

$$\begin{aligned} \frac{-(x-\mu)^2}{\sigma^2} + 1 &= 0 \\ \frac{(x-\mu)^2}{\sigma^2} &= 1 \\ (x-\mu)^2 &= \sigma^2 \\ x - \mu &= \pm\sigma \\ x &= \mu \pm \sigma. \end{aligned}$$

In other words, the graph of $n(x)$ has inflection points when $x = \mu \pm \sigma$. As we observed in the course notes, the height of these inflection points is always around 60% of the height of the maximum. This gives the "bell curve" its distinctive shape: