Math 224 Q **Exam 3A**
Fall 2017 **Tues Dec 12**

Version A

Problem 1. Let X be the continuous random variable defined by the following pdf:

$$f(x) = \begin{cases} 1 - x/2 & \text{when } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the mean $\mu = E[X]$.

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^2 x(1 - x/2)\,dx \\ &= \int_0^2 (x - x^2/2)\,dx = (x^2/2 - x^3/6)\Big|_0^2 = 4/2 - 8/6 = \boxed{2/3.} \end{aligned}$$

(b) Compute the variance $\sigma^2 = \text{Var}(X) = E[X^2] - E[X]^2$.

First we compute the second moment:

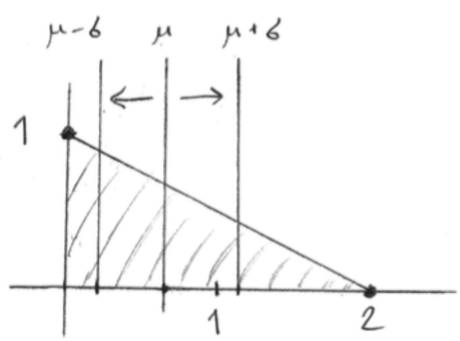
$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^2 x^2(1 - x/2)\,dx \\ &= \int_0^2 (x^2 - x^3/2)\,dx = (x^3/3 - x^4/8)\Big|_0^2 = 8/3 - 16/8 = 2/3. \end{aligned}$$

Then we have:

$$\text{Var}(X) = E[X^2] - E[X]^2 = (2/3) - (2/3)^2 = \boxed{2/9.}$$

(c) Draw the graph of $f(x)$, labeled with the mean μ and standard deviation σ.

From parts (a) and (b) we have $\mu = 2/3 = 0.67$ and $\sigma = \sqrt{2/9} = 0.47$.
Here is the picture:



Problem 2. Let $X_1, X_2, \ldots, X_{16}$ be independent samples from an underlying distribution with mean $\mu = 10$ and standard deviation $\sigma = 8$. Consider the sample mean:

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_{16}}{16}.$$

(a) Compute the expected value of the sample mean: $E[\overline{X}]$.

Since expected value is linear we have

$$E[\overline{X}] = \frac{E[X_1] + E[X_2] + \cdots + E[X_{16}]}{16} = \frac{10 + 10 + \cdots + 10}{16} = \boxed{10.}$$

(b) Compute the variance of the sample mean: $\text{Var}(\overline{X})$.

Since the observations X_i are independent we have

$$\text{Var}(\overline{X}) = \frac{\text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_{16})}{16^2} = \frac{8^2 + 8^2 + \cdots + 8^2}{16^2} = \boxed{4.}$$

(c) Assuming that $n = 16$ is large enough, use the Central Limit Theorem to estimate the probability $P(8 < \overline{X} < 11)$.

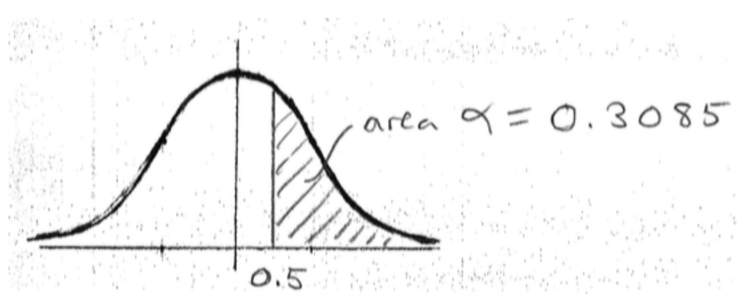
Assuming that $\overline{X}$ is approximately normal, we find that $(\overline{X} - E[\overline{X}])/\sqrt{\text{Var}(\overline{X})} = (\overline{X} - 10)/2$ is approximately standard normal. Hence

$$\begin{aligned} P(8 < \overline{X} < 11) = P(-2 < \overline{X} - 10 < 1) &= P\left(-1 < \frac{\overline{X} - 10}{2} < 0.5\right) \\ &\approx \Phi(0.5) - \Phi(-1) = \Phi(0.5) - [1 - \Phi(1)] \\ &= 0.6915 - (1 - 0.8413) = \boxed{53.28\%.} \end{aligned}$$

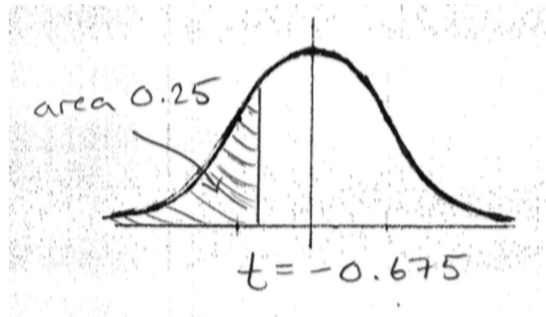
[Remark: I treated the underlying distribution as continuous.]

Problem 3. Suppose that Z is a **standard normal** random variable. Use the provided tables to solve the following problems.

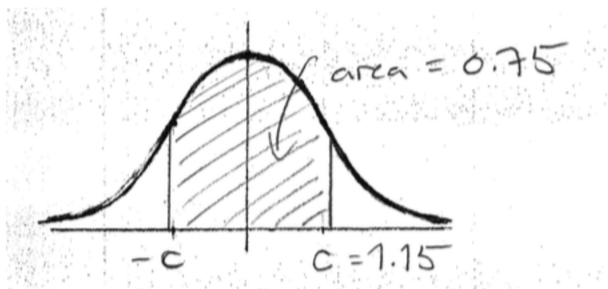
(a) Find α such that $P(Z \geq 0.5) = \alpha$ and draw a picture to illustrate your answer.



(b) Find t such that $P(Z \leq t) = 0.25$ and draw a picture to illustrate your answer.



(c) Find c such that $P(-c \leq Z \leq c) = 0.75$ and draw a picture to illustrate your answer.



Problem 4. Flip a fair coin 20 times and let X be the number of heads that you get.

(a) Write a formula for the **exact value** of $P(X \leq 12)$.

$$P(X \leq 12) = \sum_{k=0}^{12} \binom{20}{k} / 2^{20}.$$

[Remark: My laptop evaluates this to 86.84%.]

(b) Since np and $n(1-p)$ are both ≥ 10 we can assume that X is approximately normal. Use a continuity correction to approximate the probability from part (a) by the integral of some function:

$$P(X \leq 12) \approx \int_{-0.5}^{12.5} \frac{1}{\sqrt{10\pi}} e^{-(x-10)^2/10} \, dx.$$

[Remark: The lower limit $-\infty$ gives the same answer to many decimal places.]

(c) Use the provided tables to compute the value of this integral.

Note that X is binomial with $n = 20$ and $p = 1/2$, hence with $\mu = np = 10$ and $\sigma^2 = np(1-p) = 5$. Let X' be a continuous random variable with the same mean and standard deviation so that $(X' - 10)/\sqrt{5}$ is standard normal. Then we have

$$\begin{aligned} P(X \leq 12) &\approx P(-\infty \leq X' \leq 12.5) \\ &= P\left(\frac{X' - 10}{\sqrt{5}} \leq \frac{12.5 - 10}{\sqrt{5}}\right) = P\left(\frac{X' - 10}{\sqrt{5}} \leq 1.12\right) \\ &= \Phi(1.12) = \boxed{86.86\%.} \end{aligned}$$

Problem 5. Suppose that the following five independent observations come from a **normal** distribution with mean μ and variance σ^2:

1.2	1.5	2.1	2.6	1.6

(a) Compute the sample mean $\overline{X}$ and sample standard deviation S.

$$\begin{aligned} \overline{X} &= \frac{1.2 + 1.5 + 2.1 + 2.6 + 1.6}{5} = \boxed{1.8,} \\ S^2 &= \frac{(1.2 - 1.8)^2 + (1.5 - 1.8)^2 + (2.1 - 1.8)^2 + (2.6 - 1.8)^2 + (1.6 - 1.8)^2}{4} = 0.305, \\ S &= \sqrt{0.305} = \boxed{0.5522.} \end{aligned}$$

(b) Suppose for some reason that you **know** the population standard deviation $\sigma = 0.5$. In this case, find an exact 95% confidence interval for the unknown μ.

$$\overline{X} \pm z_{0.05/2} \cdot \frac{\sigma}{\sqrt{n}} = 1.8 \pm 1.96 \cdot \frac{0.5}{\sqrt{5}} = \boxed{1.8 \pm 0.438.}$$

(c) Now suppose that the population standard deviation σ is **unknown**. In this case, find an exact 95% confidence interval for the unknown μ.

$$\overline{X} \pm t_{0.05/2}(n-1) \cdot \frac{S}{\sqrt{n}} = 1.8 \pm 2.776 \cdot \frac{0.5522}{\sqrt{5}} = \boxed{1.8 \pm 0.686.}$$

Version B

Problem 1. Let X be the continuous random variable defined by the following pdf:

$$f(x) = \begin{cases} x/2 & \text{when } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the mean $\mu = E[X]$.

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^2 x(x/2)\,dx \\ &= \int_0^2 x^3/6\,dx = (x^2/6)\Big|_0^2 = 8/6 = \boxed{4/3.} \end{aligned}$$

(b) Compute the variance $\sigma^2 = \mathrm{Var}(X) = E[X^2] - E[X]^2$.

First we compute the second moment:

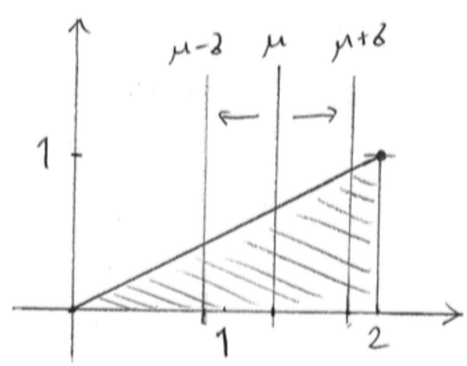
$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^2 x^2(x/2)\,dx \\ &= \int_0^2 x^3/2\,dx = (x^4/8)\Big|_0^2 = 16/8 = 2. \end{aligned}$$

Then we have:

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = 2 - (4/3)^2 = \boxed{2/9.}$$

(c) Draw the graph of $f(x)$, labeled with the mean μ and standard deviation σ.

From parts (a) and (b) we have $\mu = 4/3 = 1.33$ and $\sigma = \sqrt{2/9} = 0.47$.
Here is the picture:



Problem 2. Let $X_1, X_2, \ldots, X_{16}$ be independent samples from an underlying distribution with mean $\mu = 12$ and standard deviation $\sigma = 8$. Consider the sample mean:

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_{16}}{16}.$$

(a) Compute the expected value of the sample mean: $E[\overline{X}]$.

Since expected value is linear we have

$$E[\overline{X}] = \frac{E[X_1] + E[X_2] + \cdots + E[X_{16}]}{16} = \frac{12 + 12 + \cdots + 12}{16} = \boxed{12.}$$

(b) Compute the variance of the sample mean: $\mathrm{Var}(\overline{X})$.

Since the observations X_i are independent we have

$$\mathrm{Var}(\overline{X}) = \frac{\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_{16})}{16^2} = \frac{8^2 + 8^2 + \cdots + 8^2}{16^2} = \boxed{4.}$$

(c) Assuming that $n = 16$ is large enough, use the Central Limit Theorem to estimate the probability $P(11 < \overline{X} < 13)$.

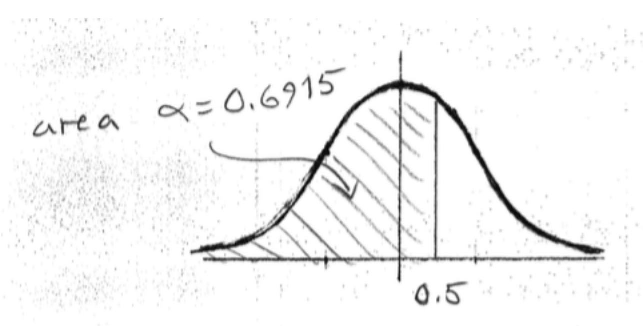
Assuming that $\overline{X}$ is approximately normal, we find that $(\overline{X} - E[\overline{X}])/\sqrt{\mathrm{Var}(\overline{X})} = (\overline{X} - 12)/2$ is approximately standard normal. Hence

$$\begin{aligned} P(11 < \overline{X} < 13) = P(-1 < \overline{X} - 12 < 1) &= P\left(-0.5 < \frac{\overline{X} - 12}{2} < 0.5\right) \\ &\approx \Phi(0.5) - \Phi(-0.5) = \Phi(0.5) - [1 - \Phi(0.5)] \\ &= 0.6915 - (1 - 0.6915) = \boxed{38.30\%.} \end{aligned}$$

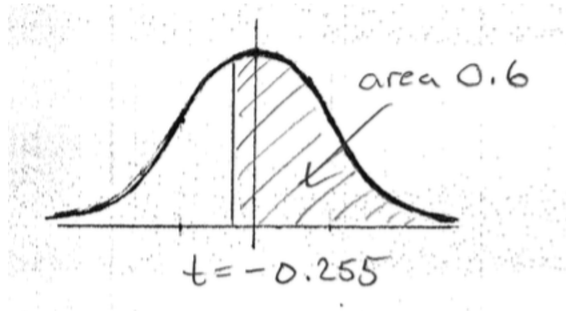
[Remark: I treated the underlying distribution as continuous.]

Problem 3. Suppose that Z is a **standard normal** random variable. Use the provided tables to solve the following problems.

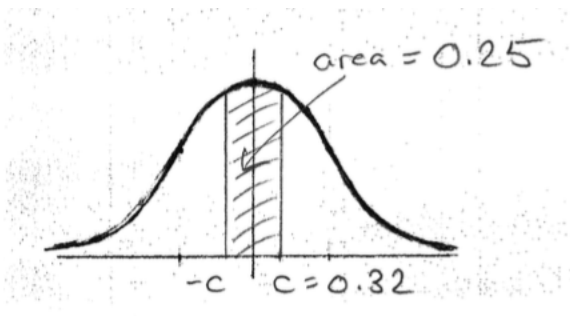
(a) Find α such that $P(Z \leq 0.5) = \alpha$ and draw a picture to illustrate your answer.



(b) Find t such that $P(Z \geq t) = 0.6$ and draw a picture to illustrate your answer.



(c) Find c such that $P(-c \leq Z \leq c) = 0.25$ and draw a picture to illustrate your answer.



Problem 4. Flip a fair coin 20 times and let X be the number of heads that you get.

(a) Write a formula for the **exact value** of $P(X \geq 9)$.

$$P(X \geq 9) = \sum_{k=9}^{20} \binom{20}{k} / 2^{20}.$$

[Remark: My laptop evaluates this to 74.83%.]

(b) Since np and $n(1-p)$ are both ≥ 10 we can assume that X is approximately normal. Use a continuity correction to approximate the probability from part (a) by the integral of some function:

$$P(X \geq 9) \approx \int_{8.5}^{20.5} \frac{1}{\sqrt{10\pi}} e^{-(x-10)^2/10} \, dx.$$

[Remark: The upper limit ∞ gives the same answer to many decimal places.]

(c) Use the provided tables to compute the value of this integral.

Note that X is binomial with $n = 20$ and $p = 1/2$, hence with $\mu = np = 10$ and $\sigma^2 = np(1-p) = 5$. Let X' be a continuous random variable with the same mean and standard deviation so that $(X' - 10)/\sqrt{5}$ is standard normal. Then we have

$$\begin{aligned} P(X \geq 9) &\approx P(\infty \geq X' \geq 8.5) \\ &= P\left(\frac{X'-10}{\sqrt{5}} \geq \frac{8.5-10}{\sqrt{5}}\right) = P\left(\frac{X'-10}{\sqrt{5}} \geq -0.67\right) \\ &= 1 - \Phi(-0.67) = \Phi(0.67) = \boxed{74.86\%.} \end{aligned}$$

Problem 5. Suppose that the following five independent observations come from a **normal** distribution with mean μ and variance σ^2:

1.4	1.5	2.1	2.4	1.6

(a) Compute the sample mean $\overline{X}$ and sample standard deviation S.

$$\begin{aligned} \overline{X} &= \frac{1.4 + 1.5 + 2.1 + 2.4 + 1.6}{5} = \boxed{1.8,} \\ S^2 &= \frac{(1.4-1.8)^2 + (1.5-1.8)^2 + (2.1-1.8)^2 + (2.4-1.8)^2 + (1.6-1.8)^2}{4} = 0.185, \\ S &= \sqrt{0.185} = \boxed{0.4301.} \end{aligned}$$

(b) Suppose for some reason that you **know** the population standard deviation $\sigma = 0.5$. In this case, find an exact 95% confidence interval for the unknown μ.

$$\overline{X} \pm z_{0.05/2} \cdot \frac{\sigma}{\sqrt{n}} = 1.8 \pm 1.96 \cdot \frac{0.5}{\sqrt{5}} = \boxed{1.8 \pm 0.438.}$$

(c) Now suppose that the population standard deviation σ is **unknown**. In this case, find an exact 95% confidence interval for the unknown μ.

$$\overline{X} \pm t_{0.05/2}(n-1) \cdot \frac{S}{\sqrt{n}} = 1.8 \pm 2.776 \cdot \frac{0.4301}{\sqrt{5}} = \boxed{1.8 \pm 0.534.}$$