Math 224 Q **Exam 2**
Fall 2017 **Tues Nov 7**

Version A

Problem 1. A Bernoulli random variable B is defined by the following table:

k	0	1
$P(B=k)$	$1-p$	p

(a) Compute the expected value $E[B]$.

The definition of expected value gives

$$E[B] = 0P(B=0) + 1P(B=1) = 0(1-p) + 1p = \boxed{p.}$$

(b) Compute the variance $\text{Var}(B)$.

We compute the second moment of B and then compute the variance:

$$E[B^2] = 0^2P(B=0) + 1^2P(B=1) = 0^2(1-p) + 1^2p = p,$$
$$\text{Var}(B) = E[B^2] - E[B]^2 = p - p^2 = \boxed{p(1-p)}.$$

(c) Let $X = X_1 + X_2 + \cdots + X_n$ where the random variables X_i are **independent** and each X_i has the same distribution as B. Compute the expected value and variance.

We use the linearity of expectation to compute

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = p + p + \cdots + p = \boxed{np.}$$

Since the X_i are independent we also have

$$\begin{aligned}\text{Var}(X) &= \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n)\\ &= p(1-p) + p(1-p) + \cdots + p(1-p) = \boxed{np(1-p).}\end{aligned}$$

Problem 2. There are 2 **blue** balls and 3 **red** balls in an urn. Suppose you reach in and grab 2 balls at random. Let X be the number of **blue** balls that you get.

(a) Compute the probability mass function of X. (Use the back of the page for rough work if necessary.)

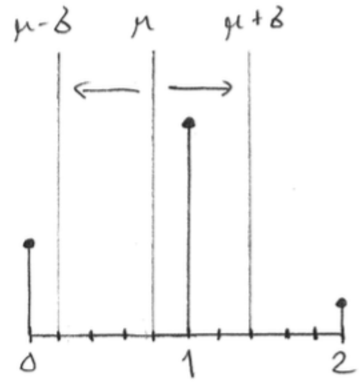
k	0	1	2
$P(X=k)$	$\frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}$	$\frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{6}{10}$	$\frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{1}{10}$

(b) Compute the mean and standard deviation of X.

We compute the first two moments, then the variance and standard deviation:

$$\begin{aligned}
E[X] &= 0\cdot\frac{3}{10}+1\cdot\frac{6}{10}+2\cdot\frac{1}{10} = 8/10 = \boxed{4/5,} \\
E[X^2] &= 0^2\cdot\frac{3}{10}+1^2\cdot\frac{6}{10}+2^2\cdot\frac{1}{10} = 10/10 = 1, \\
\mathrm{Var}(X) &= E[X^2]-E[X]^2 = 1-(4/5)^2 = 25/25-16/25 = 9/25, \\
\sigma &= \sqrt{\mathrm{Var}(X)} = \sqrt{9/25} = \boxed{3/5.}
\end{aligned}$$

(c) Draw a picture of the pmf, showing the mean and standard deviation.



Problem 3. Let X and Y be binomial random variables satisfying

$$P(X=k) = \binom{3}{k}\left(\frac{1}{2}\right)^k\left(1-\frac{1}{2}\right)^{3-k} \quad \text{and} \quad P(Y=\ell) = \binom{4}{\ell}\left(\frac{1}{3}\right)^\ell\left(1-\frac{1}{3}\right)^{4-\ell}.$$

Assume that X and Y are **independent**.

(a) Tell me the value of $E[X+Y]$.

We know (for example, from Problem 1) that the expected value of a binomial random variable is np. Then by linearity we have

$$E[X+Y] = E[X]+E[Y] = 3(1/2)+4(1/3) = 3/2+4/3 = \boxed{17/6.}$$

(b) Tell me the value of $\mathrm{Var}(X+Y)$.

We know (for example, from Problem 1) that the variance of a binomial random variable is $np(1-p)$. Then since X and Y are independent we have

$$\begin{aligned}
\mathrm{Var}(X+Y) &= \mathrm{Var}(X)+\mathrm{Var}(Y) \\
&= 3(1/2)(1/2)+4(1/3)(2/3) = 3/4+8/9 = \boxed{59/36.}
\end{aligned}$$

(c) Tell me the value of $E[(X+Y)^2]$.

There are many ways to do this. Here's one way:

$$
\begin{aligned}
E[(X+Y)^2] - E[X+Y]^2 &= \mathrm{Var}(X+Y) \\
E[(X+Y)^2] &= \mathrm{Var}(X+Y) + E[X+Y]^2 \\
&= 59/36 + (17/6)^2 = 348/36 = \boxed{29/3.}
\end{aligned}
$$

Problem 4. Consider a coin with $P(\text{heads}) = 1/3$. Start flipping the coin and let X be the number of flips until you see the first head.

(a) Compute $P(X = 5)$.

The pmf of this geometric random variable is $P(X = k) = (1-p)^{k-1}p = (2/3)^{k-1}(1/3) = 2^{k-1}/3^k$. Plugging in $k = 5$ gives

$$P(X = 5) = 2^4/3^5 = \boxed{16/243} \approx 6.58\%.$$

(b) Compute $P(X > 5)$.

By using the geometric series, one can show that $P(X > k) = (1-p)^k = (2/3)^k$. Plugging in $k = 5$ gives

$$P(X > 5) = (2/3)^5 = \boxed{32/243} \approx 13.17\%.$$

(c) Find the probability that X is within one standard deviation of its mean. [Hint: $\mu = 1/p$ and $\sigma^2 = (1-p)/p^2$.]

The mean and standard deviation are

$$\mu = 1/p = 3 \quad \text{and} \quad \sigma = (\sqrt{1-p})/p = \sqrt{2/3} \cdot 3 = \sqrt{6}.$$

Since $\sqrt{6}$ is between 2 and 3 we have

$$
\begin{aligned}
P(\mu - \sigma < X < \mu + \sigma) &= P(0 \leq X \leq 5) \\
&= 1 - P(X > 5) = 1 - 32/243 = \boxed{211/243} \approx 86.83\%.
\end{aligned}
$$

Problem 5. Let X, Y be random variables with joint pmf given by the following table:

$X \setminus Y$	-1	0	1
1	1/10	1/10	2/10
2	2/10	1/10	3/10

(a) Compute the probability mass function of X:

k	1	2
$P(X=k)$	$\frac{4}{10}=\frac{2}{5}$	$\frac{6}{10}=\frac{3}{5}$

(b) Compute the expected value $E[X]$ and the variance $\mathrm{Var}(X)$.

$$E[X] = 1\cdot\frac{2}{5} + 2\cdot\frac{3}{5} = \boxed{8/5,}$$
$$E[X^2] = 1^2\cdot\frac{2}{5} + 2^2\cdot\frac{3}{5} = 14/5,$$
$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = 14/5 - (8/5)^2 = \boxed{6/25.}$$

(c) Compute the probability mass function of Y:

k	-1	0	1
$P(Y=k)$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{5}{10}$

(d) Compute the expected value $E[Y]$ and the variance $\mathrm{Var}(Y)$.

$$E[Y] = -1\cdot\frac{3}{10} + 0\cdot\frac{2}{10} + 1\cdot\frac{5}{10} = 2/10 = \boxed{1/5,}$$
$$E[Y^2] = (-1)^2\cdot\frac{3}{10} + 0^2\cdot\frac{2}{10} + 1^2\cdot\frac{5}{10} = 8/10 = 4/5,$$
$$\mathrm{Var}(Y) = E[Y^2] - E[Y]^2 = 4/5 - (1/5)^2 = \boxed{19/25.}$$

(e) Compute the probability mass function of $Z = XY$:

k	-2	-1	0	1	2
$P(Z=k)$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$

(f) Compute the expected value $E[Z] = E[XY]$ and the covariance $\mathrm{Cov}(X,Y)$.

$$E[XY] = -2\cdot\frac{2}{10} - 1\cdot\frac{1}{10} + 0\cdot\frac{2}{10} + 1\cdot\frac{2}{10} + 2\cdot\frac{3}{10} = \boxed{3/10,}$$
$$\mathrm{Cov}(X,Y) = E[XY] - E[X]\cdot E[Y] = (3/10) - (8/5)(1/5) = \boxed{-1/50.}$$

Version B

Problem 1. Same as version A.

Problem 2. There are 3 **blue** balls and 2 **red** balls in an urn. Suppose you reach in and grab 2 balls at random. Let X be the number of **blue** balls that you get.

(a) Compute the probability mass function of X. (Use the back of the page for rough work if necessary.)

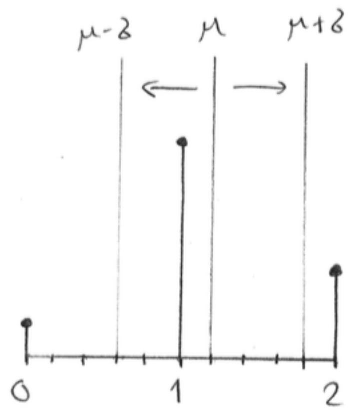
k	0	1	2
$P(X=k)$	$\frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = \frac{1}{10}$	$\frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{6}{10}$	$\frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \frac{3}{10}$

(b) Compute the mean and standard deviation of X.

We compute the first two moments, then the variance and standard deviation:

$$\begin{aligned} E[X] &= 0 \cdot \frac{1}{10} + 1 \cdot \frac{6}{10} + 2 \cdot \frac{3}{10} = 12/10 = \boxed{6/5,} \\ E[X^2] &= 0^2 \cdot \frac{1}{10} + 1^2 \cdot \frac{6}{10} + 2^2 \cdot \frac{3}{10} = 18/10 = 9/5, \\ \mathrm{Var}(X) &= E[X^2] - E[X]^2 = 9/5 - (6/5)^2 = 45/25 - 36/25 = 9/25, \\ \sigma &= \sqrt{\mathrm{Var}(X)} = \sqrt{9/25} = \boxed{3/5.} \end{aligned}$$

(c) Draw a picture of the pmf, showing the mean and standard deviation.



Problem 3. Let X and Y be binomial random variables satisfying

$$P(X=k) = \binom{3}{k}\left(\frac{1}{4}\right)^k \left(1-\frac{1}{4}\right)^{3-k} \quad \text{and} \quad P(Y=\ell) = \binom{4}{\ell}\left(\frac{1}{2}\right)^\ell \left(1-\frac{1}{2}\right)^{4-\ell}.$$

Assume that X and Y are **independent**.

(a) Tell me the value of $E[X+Y]$.

We know (for example, from Problem 1) that the expected value of a binomial random variable is np. Then by linearity we have

$$E[X+Y] = E[X] + E[Y] = 3(1/4) + 4(1/2) = 3/4 + 4/2 = \boxed{11/4.}$$

(b) Tell me the value of $\text{Var}(X+Y)$.

We know (for example, from Problem 1) that the variance of a binomial random variable is $np(1-p)$. Then since X and Y are independent we have

$$\begin{aligned}\text{Var}(X+Y) &= \text{Var}(X) + \text{Var}(Y) \\ &= 3(1/4)(3/4) + 4(1/2)(1/2) = 9/16 + 4/4 = \boxed{25/16.}\end{aligned}$$

(c) Tell me the value of $E[(X+Y)^2]$.

There are many ways to do this. Here's one way:

$$\begin{aligned}E[(X+Y)^2] - E[X+Y]^2 &= \text{Var}(X+Y) \\ E[(X+Y)^2] &= \text{Var}(X+Y) + E[X+Y]^2 \\ &= 25/16 + (11/4)^2 = \boxed{73/8.}\end{aligned}$$

Problem 4. Consider a coin with $P(\text{heads}) = 2/3$. Start flipping the coin and let X be the number of flips until you see the first head.

(a) Compute $P(X = 5)$.

The pmf of this geometric random variable is $P(X = k) = (1-p)^{k-1}p = (1/3)^{k-1}(2/3) = 2/3^k$. Plugging in $k = 5$ gives

$$P(X = 5) = 2/3^5 = \boxed{2/243} \approx 0.8\%.$$

(b) Compute $P(X > 5)$.

By using the geometric series, one can show that $P(X > k) = (1-p)^k = (1/3)^k$. Plugging in $k = 5$ gives

$$P(X > 5) = (1/3)^5 = \boxed{1/243} \approx 0.4\%.$$

(c) Find the probability that X is within one standard deviation of its mean. [Hint: $\mu = 1/p$ and $\sigma^2 = (1-p)/p^2$.]

The mean and standard deviation are

$$\mu = 1/p = 3/2 \quad \text{and} \quad \sigma = (\sqrt{1-p})/p = \sqrt{1/3} \cdot 3/2 = \sqrt{3}/2.$$

Since $\sqrt{3}/2$ is between 0.5 and 1 (I'll be generous on this one) we have

$$\begin{aligned} P(\mu - \sigma < X < \mu + \sigma) &= P(X = 1) + P(X = 2) \\ &= (2/3) + (1/3)(2/3) = \boxed{8/9} \approx 88.89\%. \end{aligned}$$

Problem 5. Let X, Y be random variables with joint pmf given by the following table:

$X \setminus Y$	-1	0	1
1	1/10	2/10	1/10
2	2/10	1/10	3/10

(a) Compute the probability mass function of X:

k	1	2
$P(X = k)$	$\frac{4}{10} = \frac{2}{5}$	$\frac{6}{10} = \frac{3}{5}$

(b) Compute the expected value $E[X]$ and the variance $\mathrm{Var}(X)$.

$$\begin{aligned} E[X] &= 1 \cdot \frac{2}{5} + 2 \cdot \frac{3}{5} = \boxed{8/5,} \\ E[X^2] &= 1^2 \cdot \frac{2}{5} + 2^2 \cdot \frac{3}{5} = 14/5, \\ \mathrm{Var}(X) &= E[X^2] - E[X]^2 = 14/5 - (8/5)^2 = \boxed{6/25.} \end{aligned}$$

(c) Compute the probability mass function of Y:

k	-1	0	1
$P(Y = k)$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

(d) Compute the expected value $E[Y]$ and the variance $\mathrm{Var}(Y)$.

$$E[Y] = -1 \cdot \frac{3}{10} + 0 \cdot \frac{3}{10} + 1 \cdot \frac{4}{10} = \boxed{1/10,}$$
$$E[Y^2] = (-1)^2 \cdot \frac{3}{10} + 0^2 \cdot \frac{3}{10} + 1^2 \cdot \frac{4}{10} = 7/10,$$
$$\mathrm{Var}(Y) = E[Y^2] - E[Y]^2 = 7/10 - (1/10)^2 = \boxed{69/100.}$$

(e) Compute the probability mass function of $Z = XY$:

k	-2	-1	0	1	2
$P(Z = k)$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{3}{10}$

(f) Compute the expected value $E[Z] = E[XY]$ and the covariance $\mathrm{Cov}(X, Y)$.

$$E[XY] = -2 \cdot \frac{2}{10} - 1 \cdot \frac{1}{10} + 0 \cdot \frac{3}{10} + 1 \cdot \frac{1}{10} + 2 \cdot \frac{3}{10} = 2/10 = \boxed{1/5,}$$
$$\mathrm{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y] = (2/10) - (8/5)(1/10) = \boxed{1/25.}$$