This Week: Gradients.

(Chapter 4: Differentiation of multivariable functions)

A Function $f: \mathbb{R}^n \to \mathbb{R}$ is called a "scalar field". To each point $P = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ it associates a scalar

$f(P)$ or $f(x_1, x_2, \ldots, x_n)$.

Think: $f(P)$ is temperature at the point P.

The derivative of f defined as

$$\nabla f = \left\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \ldots, \frac{df}{dx_n} \right\rangle$$

$$\left(\text{or } \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \right)$$

Notation: $\frac{\partial f}{\partial x_i}$ means we take deriv. with respect to variable x_i, pretend that the other vars. are constant.

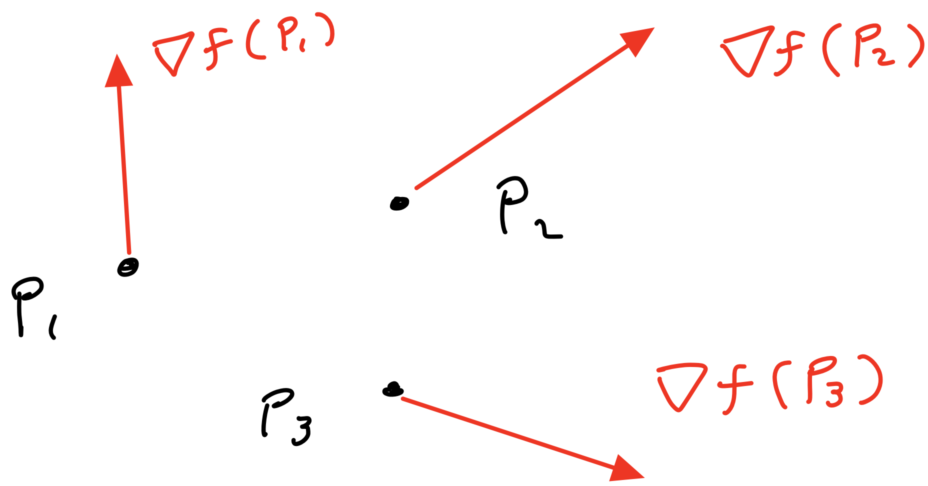
e.g. $f(x,y) = xy.$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x,y) = \langle y, x \rangle$$

To each point P in $\mathbb{R}^n$ the derivative ∇f associates a vector.

Picture:



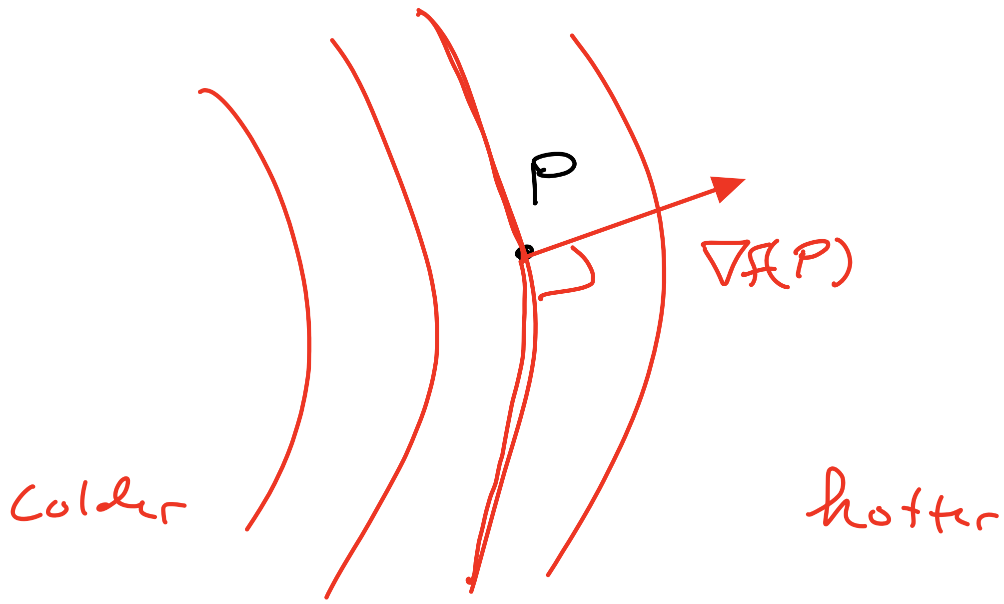
Can think of $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ as a "vector field".

So

"∇(scalar field) = vector field."

Meaning: At a point P in $\mathbb{R}^n$, the vector $\nabla f(P)$ tells us the direction where f is increasing fastest.

e.g. f is temperature



curves of constant temp.

$\nabla f(P)$ points in the direction of increasing temperature & is $\perp$ to the "level curve" at P.

e.g. $f(x,y) = xy$. defined at any point

$\nabla f(x,y) = \langle y, x \rangle$.

pick this point

Consider point $P = (3,1)$.

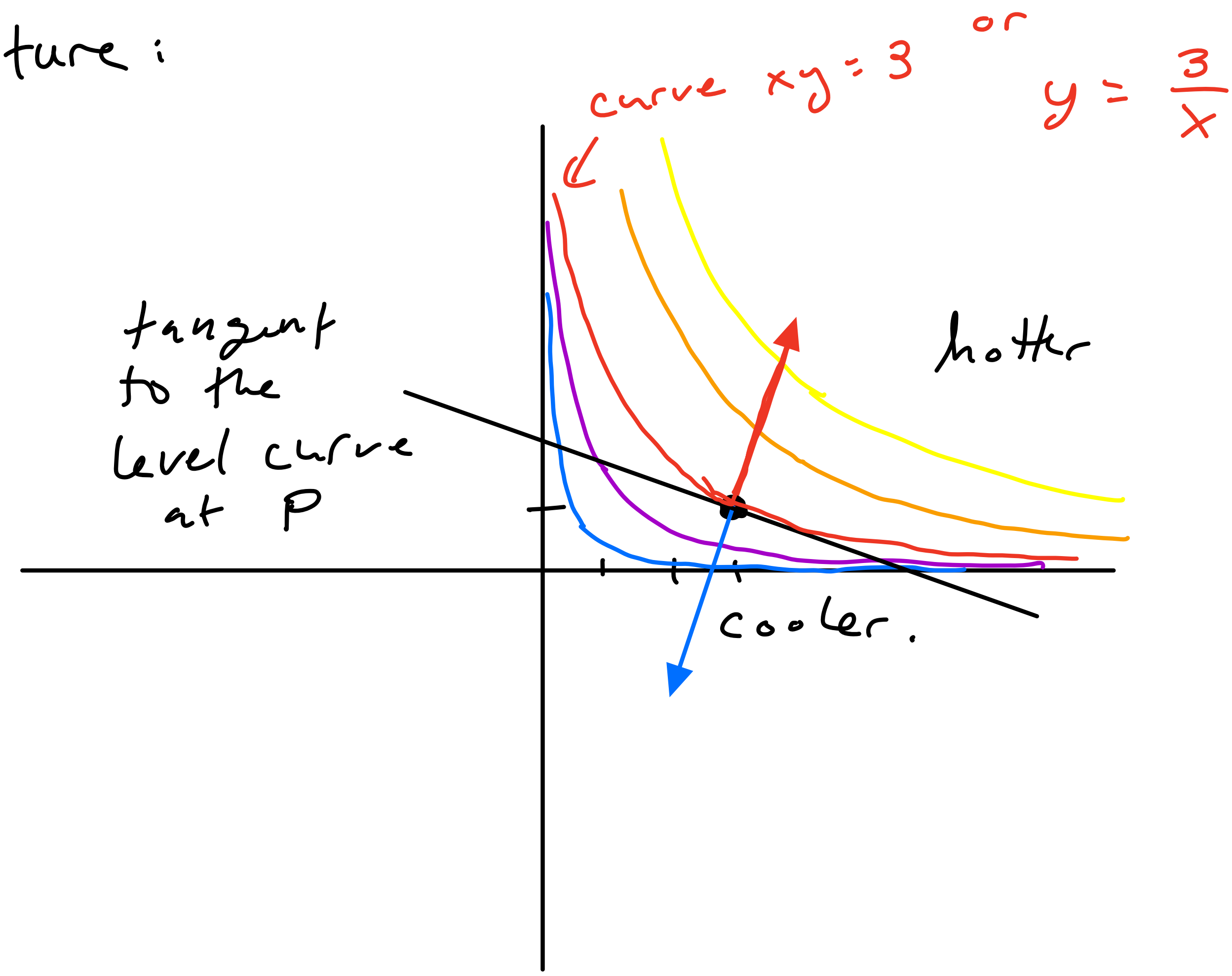
$f(P) = 3 \cdot 1 = 3$.

" temperature at P is 3 ".

Temperature increases fastest in the direction $\nabla f(P) =$

$\nabla f(3,1) = \langle 1, 3 \rangle$

Picture:



$-\nabla f(P)$ is the direction in which

f decreases most rapidly at P.

The "same story" in 3D.

Say $f(x,y,z) = 5x^2 - 3xy + xyz$

is temperature at point $P = (x,y,z)$.

$$\nabla f = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle$$

$df/dx = 10x - 3y + yz$

$df/dy = 0 - 3x + xz$

$df/dz = 0 - 0 + xy$

$\nabla f(x,y,z)$

$= \langle 10x - 3y + yz, -3x + xz, xy \rangle$.

e.g. Temperature at $P = (1,1,1)$ is

$f(1,1,1) = 5(1)^2 - 3(1)(1) + (1)(1)(1)$

$= 5 - 3 + 1 = 3$

and increases most rapidly in the direction

$$\nabla f(1,1,1) = \langle 10-3+1, -3+1, 1\rangle$$
$$= \langle 8, -2, 1\rangle.$$

Why does it work?

KEY: Multivariable Chain Rule.

Recall the good old chain rule.

$$f : \mathbb{R} \to \mathbb{R}$$
$$g : \mathbb{R} \to \mathbb{R}.$$

$$(f \circ g)(t) = f(g(t)).$$

$$(f \circ g)'(t) = f'(g(t)) \cdot g'(t)$$
$$= (f' \circ g)(t) \cdot g'(t)$$

In Chapter 3 we had a new version.

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$
$$f : \mathbb{R} \to \mathbb{R}$$

$$(\vec{r} \circ f)(t) = \vec{r}(f(t)).$$

$$(\vec{r} \circ f)'(t) = \underbrace{\vec{r}\,'(f(t))}_{\text{vector}} \cdot \underbrace{f'(t)}_{\text{scalar}}.$$

Now we have something new.

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

Say $\vec{r}(t)$ is a parameterized path in $\mathbb{R}^n$, say f is temperature. So $f(\vec{r}(t))$ is the temperature we feel at time t. We can think of this as a composition:

$$(f \circ \vec{r})(t) = f(\vec{r}(t))$$

$$f \circ \vec{r} : \mathbb{R} \to \mathbb{R}.$$
$$t \mapsto \text{our temp. at time } t$$

as we travel the curve.

Question: What temperature change do we feel?

$$(f \circ \vec{r})'(t) = \ ?$$

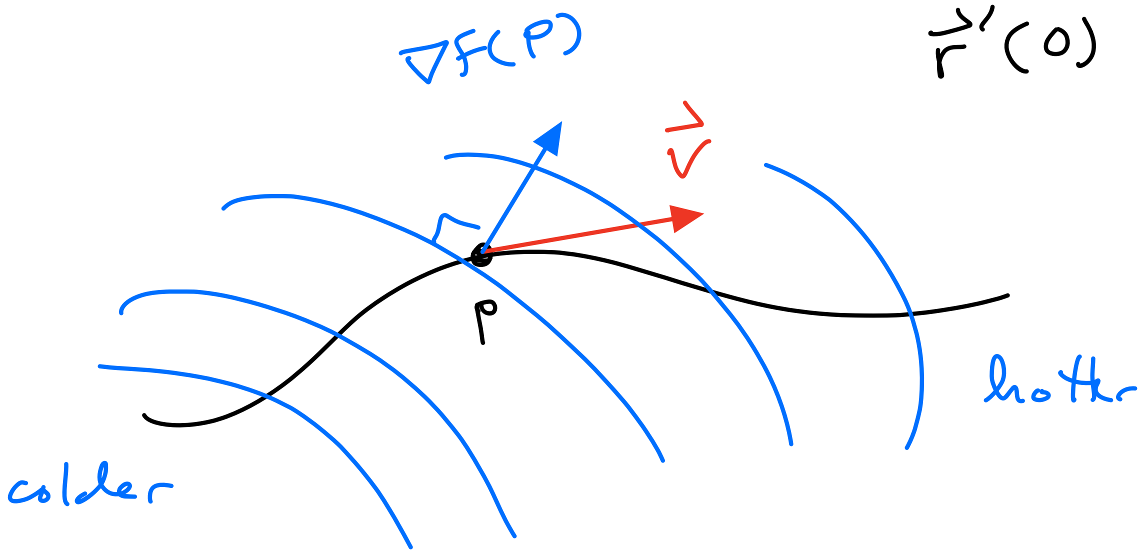
Theorem:

dot product.

$$\underbrace{(f \circ \vec{r})'(t)}_{\text{scalar}} = \underbrace{\nabla f(\vec{r}(t))}_{\text{vector}} \cdot \underbrace{\vec{r}\,'(t)}_{\text{vector}}$$

e.g. The rate of change of f at the point P in the direction of vector $\vec{v}$. Suppose $\vec{r}(0) = P$

$\vec{r}\,'(0) = \vec{v}$



Our rate of change of temp at time $t = 0$ is

$$(f \circ \vec{r})'(0)$$

$$= \nabla f(\vec{r}(0)) \cdot \vec{r}'(0)$$

$$= \nabla f(P) \cdot \vec{v}.$$

$$= \underbrace{(\text{gradient}) \cdot (\text{velocity})}.$$

maximized when our velocity points in direction of gradient.

Another Consequence: The gradient vector at P is $\perp$ to the level set of f at P.

e.g. Consider $f(x,y,z) = (2x)^2 + y^2 + z^2$.

Temperature at point $P = (2,3,5)$ is

$f(2,3,5) = (2 \cdot 2)^2 + 3^2 + 5^2 = 50.$

Gradient vector is

$$\nabla f(x,y,z) = \langle 8x, 2y, 2z \rangle.$$

$$\nabla f(2,3,5) = \langle 16, 6, 10 \rangle.$$

The level set $f(x,y,z) = 50$

is the set of points where

temperature = 50.

$$(2x)^2 + y^2 + z^2 = 50.$$

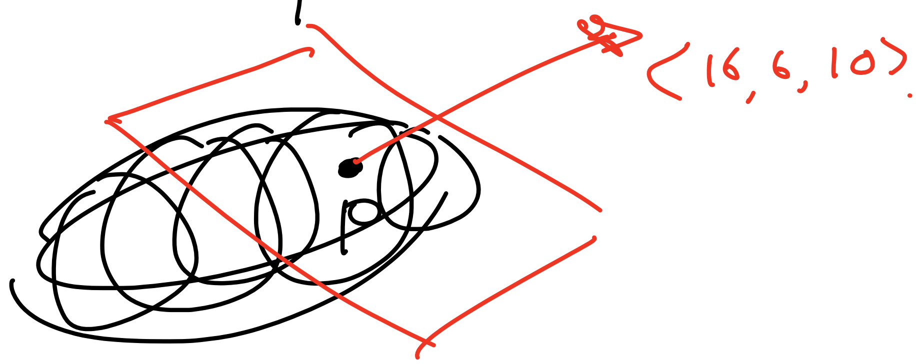
What kind of shape?

Fixed x $\rightsquigarrow$ circle in y, z.

Fixed y $\rightsquigarrow$ ellipse in x, z

Fixed z $\longrightarrow$ ellipse in x, y.

Called an "ellipsoid"



Theorem: $\langle 16, 6, 10 \rangle$ is normal to the tangent plane to the level surface at $P = (2, 3, 5)$.

Conclusion:

Equation of tangent plane to ellipsoid $(2x)^2 + y^2 + z^2 = 50$ at the point $(2, 3, 5)$ is

$$\nabla f(2,3,5) \cdot \langle x-2, y-3, z-5 \rangle = 0.$$

$$\langle 16, 6, 10 \rangle \cdot \langle x-2, y-3, z-5 \rangle = 0$$

$$16(x-2) + 6(y-3) + 10(z-5) = 0$$

$$\vdots$$

$$8x + 3y + 5z = 50.$$