HW 2 Comments :

Problem 3: If you have

$$\vec{r}(0) = \langle x_0, y_0 \rangle$$
$$\vec{v}(0) = \langle u_0, v_0 \rangle$$
$$\vec{a}(t) = \langle 0, -g \rangle$$

Then,

$$\vec{r}(t) = \left\langle x_0 + u_0 t,\ y_0 + v_0 t - \tfrac{1}{2} g t^2 \right\rangle$$
$$= \left\langle 0 + t\cos\theta,\ 0 + t\sin\theta - \tfrac{1}{2} g t^2 \right\rangle$$

HW 2 due tomorrow (Fri)

No class Mon.

Quiz 2 on Tues beginning of class.

Start Chapter 4 early.

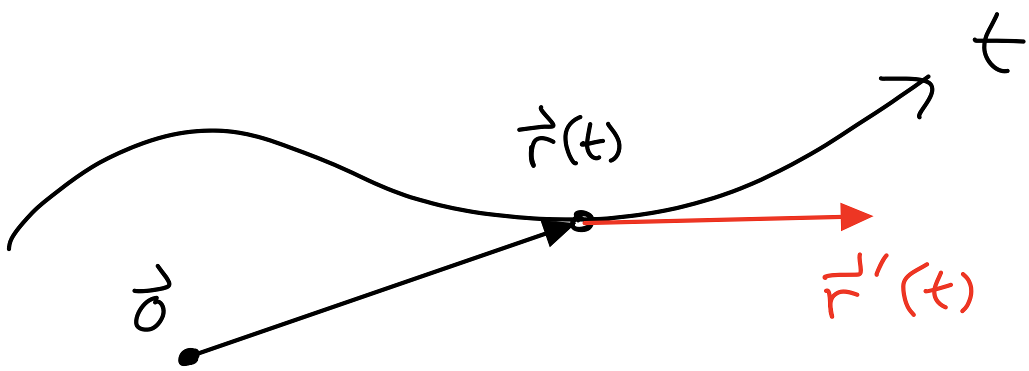
Chap 4: Differentiation of functions of several values.

(i.e. "gradient vectors").

So far we have studied functions

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

Picture: Parametrized path



The derivative is velocity:

$$\vec{r}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle$$

$$\vec{r}'(t) = \langle x_1'(t), x_2'(t), \ldots, x_n'(t) \rangle$$

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt} \right\rangle.$$

Applications:

- Integrate $\|\vec{r}'(t)\|$ to get arc length.
- Newton's 2nd Law.

$$\vec{v}(t) = \vec{r}\,'(t)$$
$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$
$$\vec{F}(t) = m\,\vec{a}(t) = m\,\vec{r}\,''(t).$$

Now we consider functions

$$f : \mathbb{R}^n \to \mathbb{R}.$$

To each point $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ we assign a number (i.e. scalar)

$$f(x_1, x_2, \dots, x_n).$$

Called a "scalar field".

e.g. temperature
pressure
density
etc.

Example: $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \left(\frac{x}{2}\right)^2 + y^2$$

We can visualize this in 2 ways:

- Level Curves:

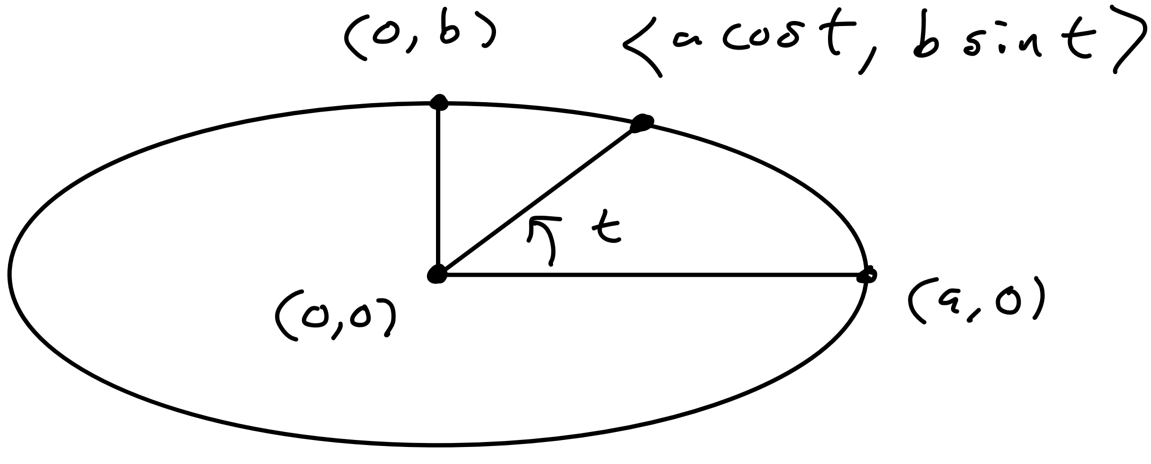
For each fixed constant $c \in \mathbb{R}$ we get a curve in $\mathbb{R}^2$ defined by

$$f(x,y) = c.$$

In our case, each level curve is an ellipse

$$\left(\frac{x}{2}\right)^2 + y^2 = c$$

[Aside: Equation of an ellipse.



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Special case: $a = b = r$

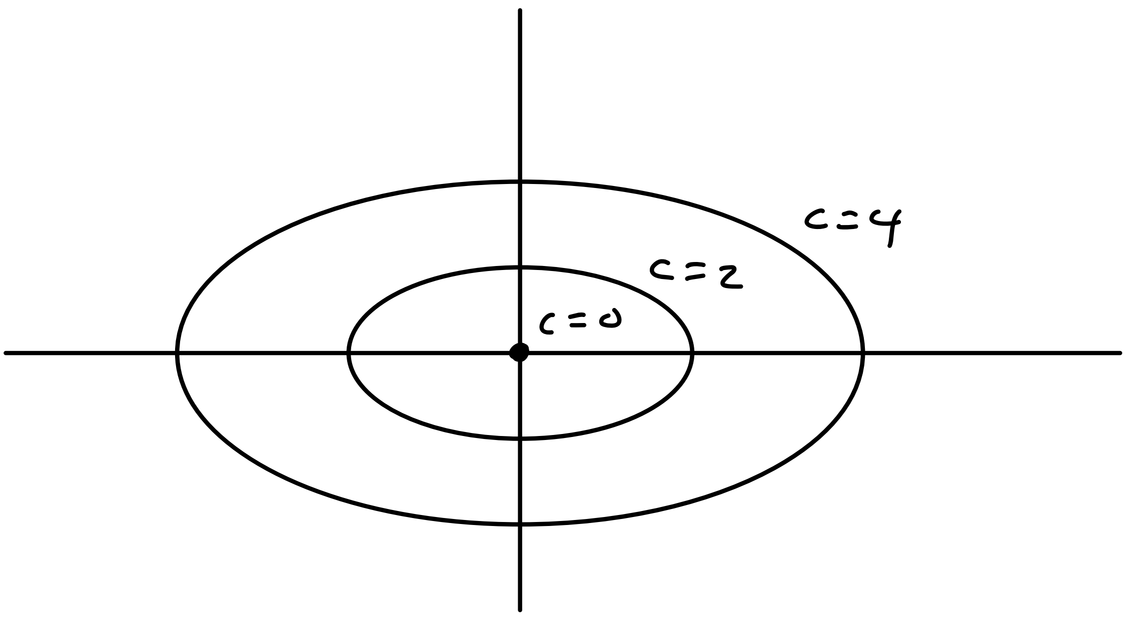
$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$x^2 + y^2 = r^2$$

circle of radius r.]

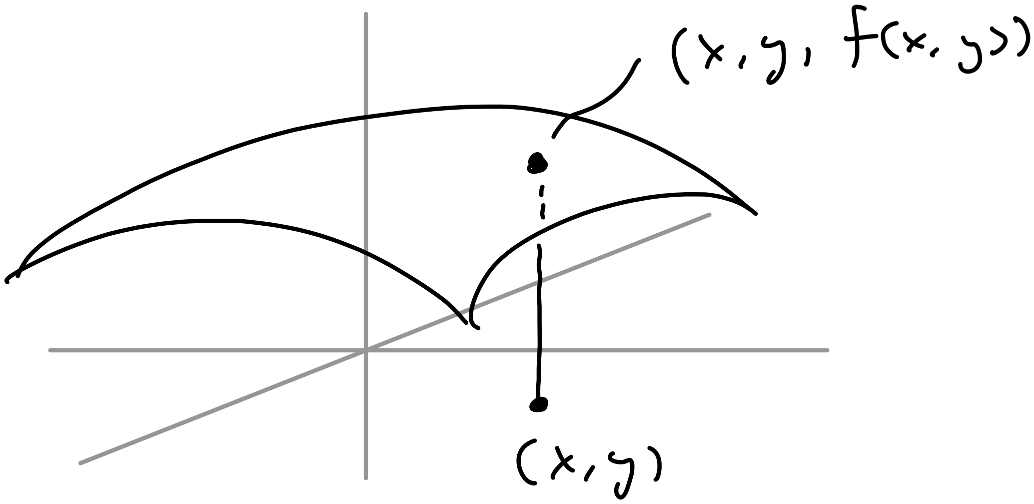
So $\left(\frac{x}{2}\right)^2 + y^2 = c$

$$\left(\frac{x}{2\sqrt{c}}\right)^2 + \left(\frac{y}{\sqrt{c}}\right)^2 = 1.$$



Infinitely many level curves. You have to imagine (or use color, heat map if c = temperature).

• We can also visualize $f(x,y)$ as the 2D surface $z = f(x,y)$.

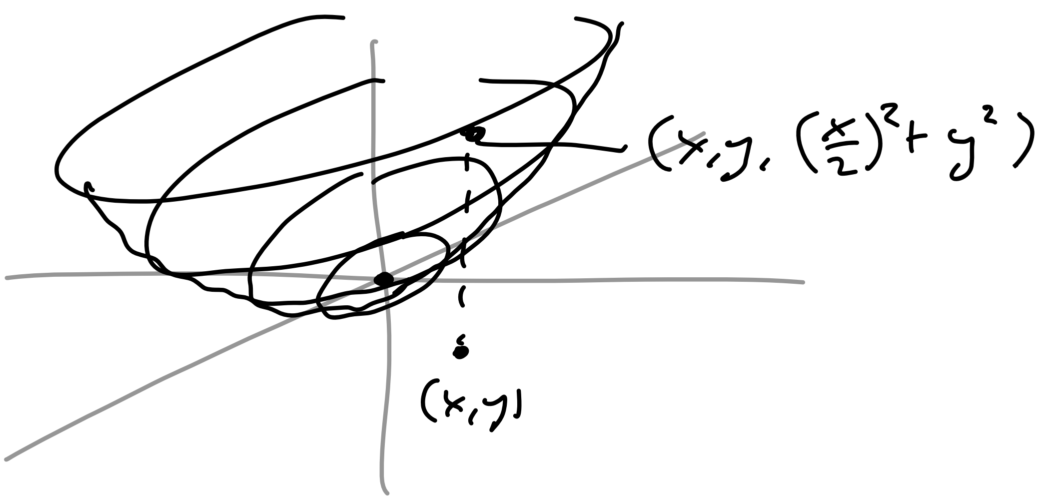


The height of the surface above the point (x,y) is the scalar $f(x,y)$.

[Think: Height = temperature ...]

Our Example: $z = \left(\frac{x}{2}\right)^2 + y^2$

is a parabolic bowl ("paraboloid")



Remark: The graph $z = f(x,y)$ is a useful visualization but it does not work for scaler fields $f : \mathbb{R}^3 \to \mathbb{R}$.

Say $f(x,y,z)$ = temperature at point (x,y,z) in $\mathbb{R}^3$.

Then the "graph" $w = f(x,y,z)$ is a "curvy 3D shape" living in 4D space, which is NOT HELPFUL. In this case our only hope is to visualize the "level surfaces"

$$f(x,y,z) = c \quad \text{for fixed } c.$$

These are the points in $\mathbb{R}^3$ with a given fixed temperature c.

BIG QUESTION:

What is the "derivative" of a scalar field $f: \mathbb{R}^n \to \mathbb{R}$?

Ideas:

(1) If f is temperature at a point then derivative should be "rate of change of temperature".

Problem: Rate of change of temp depends on our velocity.

(1)' Given a point P and a velocity $\vec{v}$, what rate of change of temperature do you feel?

(2) Recall: df/dx is the slope of the tangent line to the curve $y = f(x)$ in the plane.

So... deriv of $f(x,y)$ should be "slope of tangent plane" to the surface $z = f(x,y)$.

Problem: A plane is not determined by a slope; it is determined by a normal vector.

②′ Derivative of $f(x,y)$ should give the normal vector to the tangent plane to surface $z = f(x,y)$ at a given point.

Good News: Turns out problems ①′ & ②′ are the same, determined by the GRADIENT VECTOR.

Def: Given $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x_1, x_2, \dots, x_n)$$

we define the gradient vector

$$\nabla f = \left\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \dots, \frac{df}{dx_n} \right\rangle$$

Example: $f(x,y) = \frac{1}{4}x^2 + y^2$

$$\nabla f(x,y) = \left\langle \frac{1}{2}x, 2y \right\rangle$$

[$\frac{d}{dx}\left(\frac{1}{4}x^2 + y^2\right) = \frac{1}{4}2x + 0$

Because d/dx treats y as a constant. Most books write

$\frac{\partial}{\partial x}$ instead of $\frac{d}{dx}$

and call this a "partial derivative with respect to x".]

Remarks:

- ∇ is called "nabla".
- The vector $\nabla f(x,y)$ changes from point to point, so we can

think of ∇f as a "vector field"

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$$

Called the "gradient vector field".

[Examples of vector fields.

force fields (gravity, electric, ...)

velocity fields (wind)]

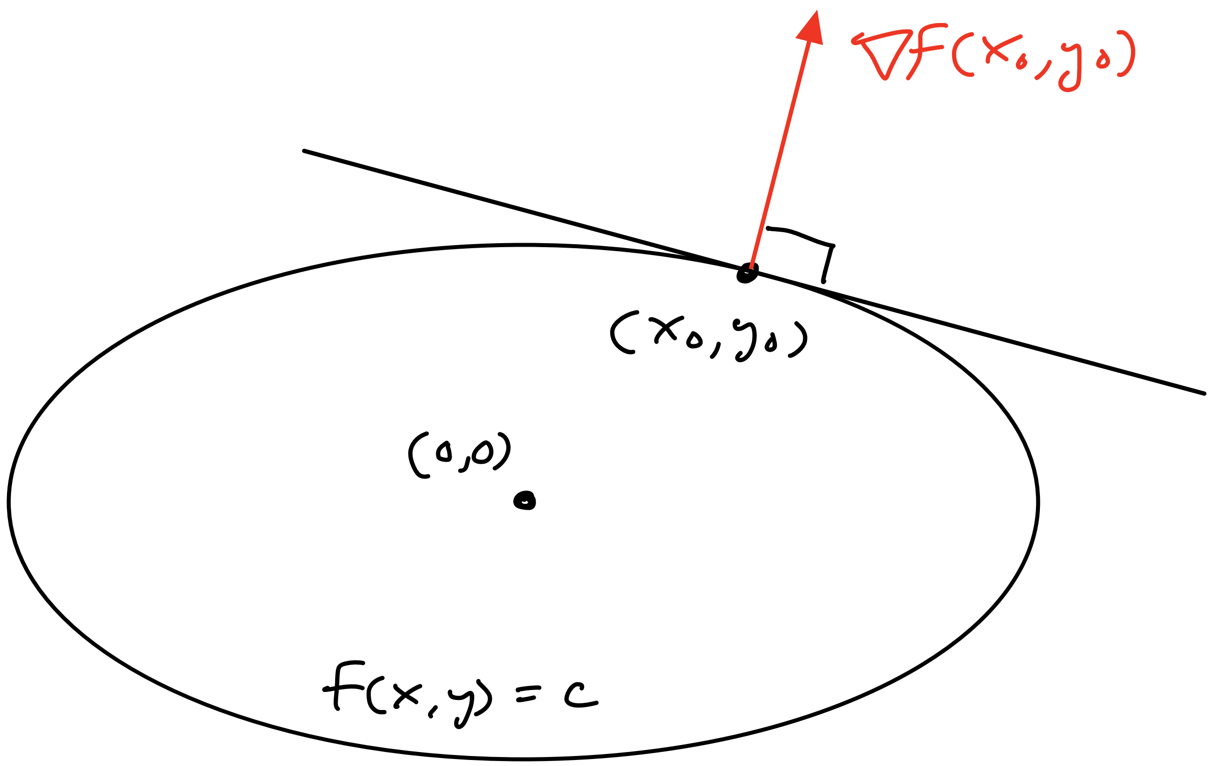
The gradient vector to

$$f(x,y) = \frac{1}{4}x^2 + y^2$$

at the point (x_0, y_0) is

$$\nabla f(x_0, y_0) = \left\langle \frac{1}{2}x_0, 2y_0 \right\rangle.$$

MEANING: If $c = f(x_0, y_0)$ is the temperature at this point then $\nabla f(x_0, y_0)$ is perpendicular to the level curve:



Meaning: $\nabla f(x_0,y_0)$ is the direction of maximum increase of temperature at the point (x_0,y_0).

Two Rigorous Statements.

- The tangent line to the curve $f(x,y) = \text{constant} = c$

at the point (x_0, y_0) is

$$\frac{dF}{dx}(x_0,y_0)(x-x_0) + \frac{dF}{dy}(x_0,y_0)(y-y_0) = 0.$$

$$\nabla f(x_0,y_0) \cdot (x-x_0, y-y_0) = 0$$

this is a normal vector

this is a point on the line.

In our case

$$\nabla f(x_0,y_0) \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$\langle \tfrac{1}{2}x_0, 2y_0 \rangle \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$\frac{1}{2}x_0(x-x_0) + 2y_0(y-y_0) = 0$$

The tangent line to the curve

$$\frac{1}{4}x^2 + y^2 = \text{constant}$$

at the point (x_0, y_0).

e.g. The tangent line to ellipse

$$\frac{1}{4}x^2 + y^2 = 1$$

at point $(\underset{x_0}{\sqrt{2}}, \underset{y_0}{\sqrt{2}/2})$ has equation

$$\frac{1}{2}x_0(x-x_0) + 2y_0(y-y_0) = 0$$

$$\frac{1}{2}\sqrt{2}(x-\sqrt{2}) + 2\frac{\sqrt{2}}{2}\left(y-\frac{\sqrt{2}}{2}\right) = 0$$

$$\frac{\sqrt{2}}{2}x - 1 + \sqrt{2}y - 1 = 0$$

$$\frac{\sqrt{2}}{2}x + \sqrt{2}y = 2$$

$$\frac{1}{2}x + y = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Example: Find equation of the tangent line to circle

$$x^2 + y^2 = 25$$

at the point $(x,y) = (3,4)$.

Let $f(x,y) = x^2 + y^2$.

Then $\nabla f(x,y) = \langle 2x, 2y \rangle$.

$\nabla f(3,4) = \langle 6, 8 \rangle$.

So tangent line is

$$\nabla f(3,4) \cdot \langle x-3, y-4 \rangle = 0$$

$$\langle 6, 8 \rangle \cdot \langle x-3, y-4 \rangle = 0$$

$$6(x-3) + 8(y-4) = 0.$$

$$6x + 8y - 18 - 32 = 0$$

$$6x + 8y = 50$$