There is no such thing as "the equation of a line" in $\mathbb{R}^3$.

Instead we need at least 2 equations.

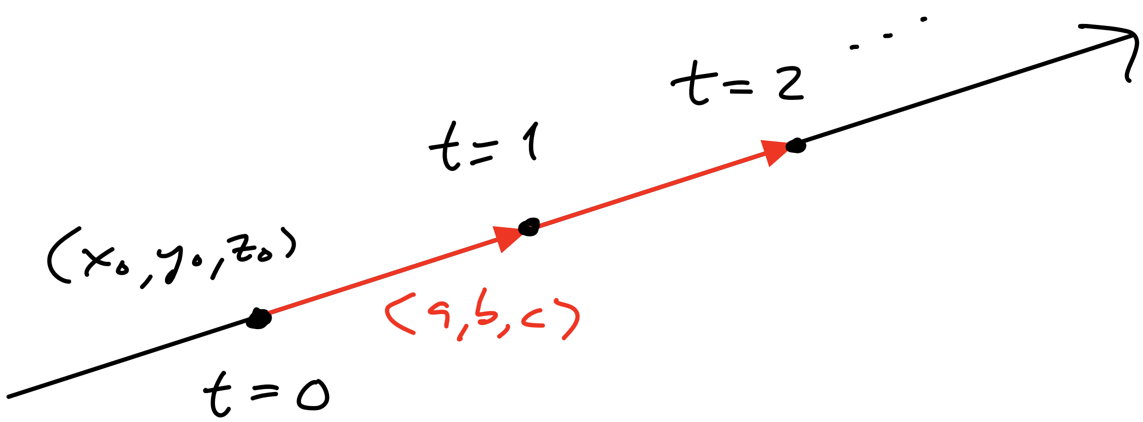
Geometrically: A line is an intersection of 2 planes in $\mathbb{R}^3$.

Two ways to describe a line in $\mathbb{R}^3$.

- Parametrization:

$$\vec{r}(t) = \vec{x}_0 + t\vec{a}$$

$$= \langle x_0 + ta, \ y_0 + tb, \ z_0 + tc \rangle$$



One parameter because a line is a "one dimensional" shape.

- As the solution of a system of 2 linear equations in 3 unknowns.

Example:

$$\begin{cases} x - y + 0 = 1, & (1) \\ x + y + 2z = 1. & (2) \end{cases}$$

To find a parametrization of the line we let $t = z$ be the parameter, then solve for x & y in terms of t. Method:

Find an equation without x and an equation without y.

$$(1): \quad x - y = 1$$

$$(2): \quad x + y = 1 - 2\overset{t}{\cancel{z}}$$

$$(1)+(2): \quad 2x + 0 = 2 - 2\overset{t}{\cancel{z}}$$

$$(2)-(1): \quad 0 + 2y = 0 - 2\overset{t}{\cancel{z}}$$

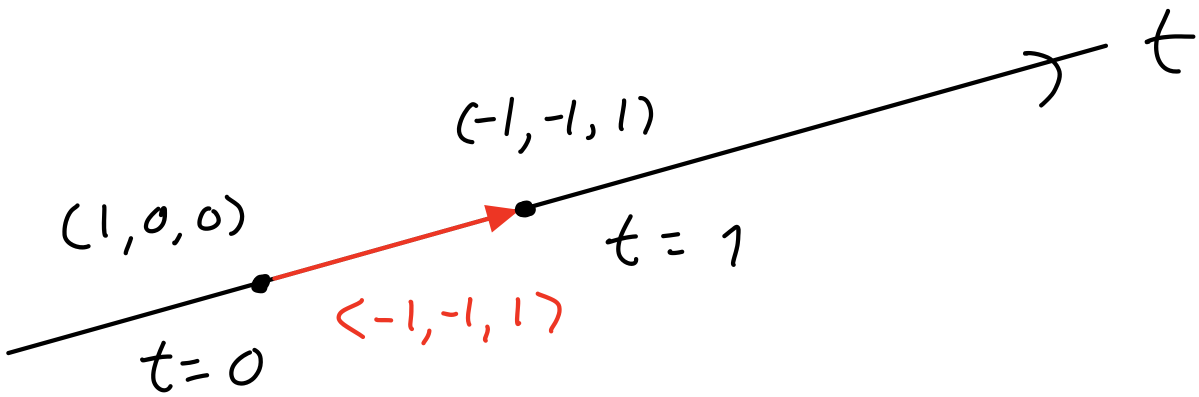
We conclude that

$$\begin{cases} x = 1 - t \\ y = -t \\ z = t \end{cases}$$

$$\vec{r}(t) = \langle x, y, z \rangle = \langle 1-t, -t, t \rangle$$

$$= \langle 1-t, 0-t, 0+t \rangle = \langle 1, 0, 0 \rangle + \langle -t, -t, t \rangle$$

$$= \langle 1, 0, 0 \rangle + t \langle -1, -1, 1 \rangle$$

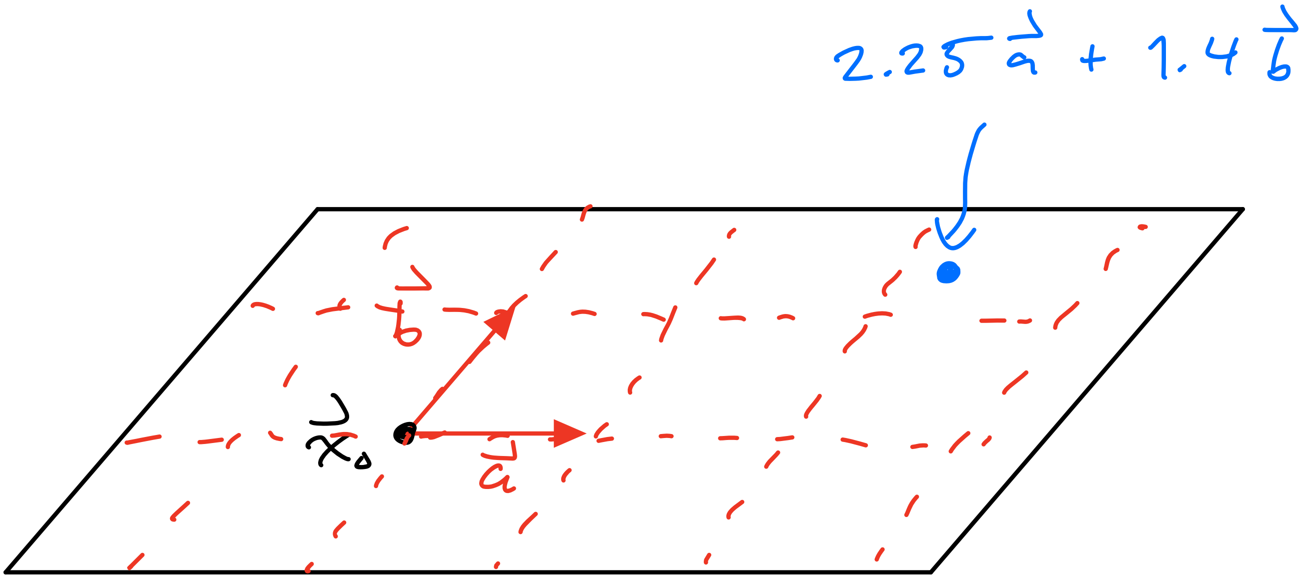


We can also give a "parametric description" of a plane in $\mathbb{R}^3$.
Since a plane is "2 dimensional" we will need 2 independent parameters.
What should we call them?

Our book uses u & v.

A parametric plane has the form

$$\vec{r}(u,v) = \vec{x}_0 + u\vec{a} + v\vec{b}$$

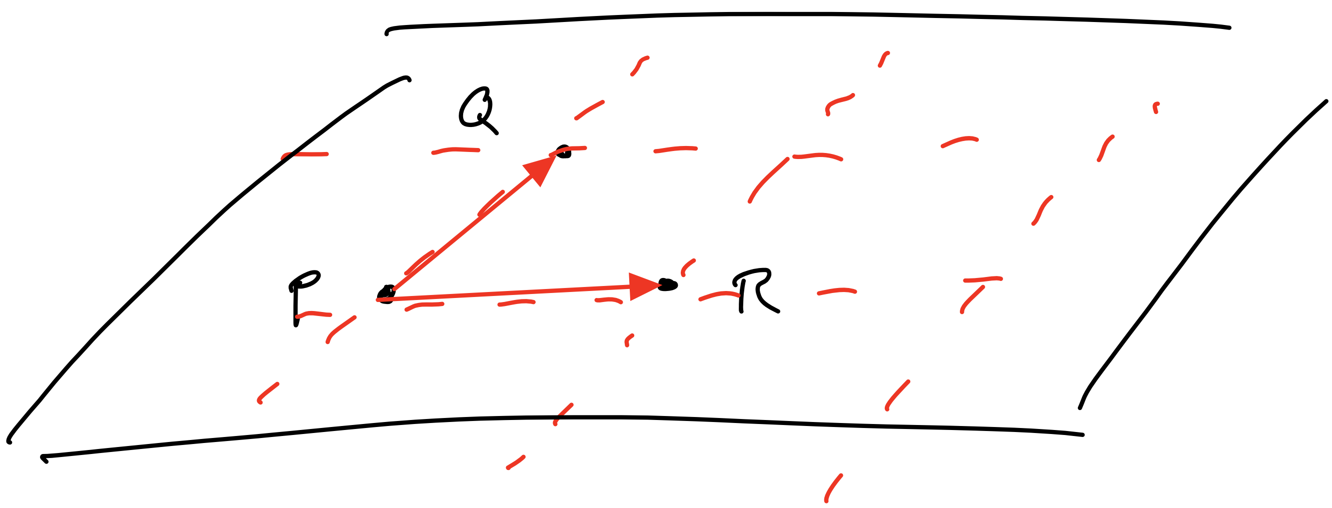


Every point in the plane has unique "coordinates" u, v with respect to the "basic vectors" $\vec{a}$ & $\vec{b}$. This is the only way to introduce a "coordinate system" on a general plane: Pick a point & pick a "basis" of two direction vectors.

Example: The plane generated by points $P = (1,1,0)$

$Q = (1,0,2)$

$R = (1,2,3)$



$$\vec{r}(u,v) = P + u\,\vec{PQ} + v\,\vec{PR}$$

$$= \langle 1,1,0\rangle + u\langle 0,-1,2\rangle + v\langle 0,2,1\rangle$$

$$= \langle 1,\ 1-u+2v,\ 0+2u+v\rangle$$

Example: Parametrize the plane

$$2x + 3y - z = 5$$

- Find 3 points?

- Easiest: Let $y = u$ & $z = v$ be parameters. Then solve for x.

$$2x + 3u - v = 5$$
$$2x = 5 - 3u + v$$
$$x = \frac{5}{2} - \frac{3}{2}u + \frac{1}{2}v$$
$$y = u$$
$$z = v$$

$$\vec{r}(u,v) = \langle \overbrace{\frac{5}{2} - \frac{3}{2}u + \frac{1}{2}v}^{x}, \overbrace{u}^{y}, \overbrace{v}^{z} \rangle$$
$$= \langle \frac{5}{2} - \frac{3}{2}u + \frac{1}{2}v, 0 + 1u + 0v, 0 + 0u + 1v \rangle$$
$$= \langle \frac{5}{2}, 0, 0 \rangle + u\langle -\frac{3}{2}, 1, 0 \rangle + v\langle \frac{1}{2}, 0, 1 \rangle.$$

Later we will parametrize general surfaces in $\mathbb{R}^3$. We need to do this so we can <u>integrate</u> over a surface.

Example: Let $\vec{r}(u,v)$ be a parametrized surface. Let

$$\vec{r}_u(u,v) = \frac{d}{du}\vec{r}(u,v)$$

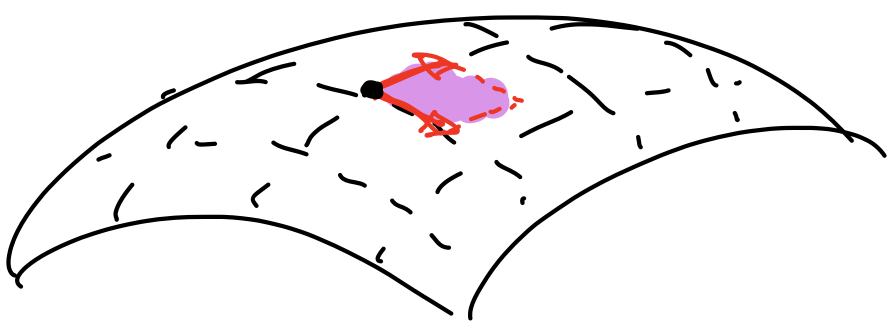
$$\vec{r}_v(u,v) = \frac{d}{dv}\vec{r}(u,v)$$

The <u>surface area</u> is a double integral

$$\iint |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

integrate over all relevant values of u, v.

Reason: $|\vec{r}_u \times \vec{r}_v| \, du \, dv$ is the area of a tiny parallelogram.



Stay Tuned.

Motion in Space & Newton's Second Law.

Consider a parametrized curve in 3D:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

Think of t as time.

At each time there is a velocity vector & an acceleration vector:

$$\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle x''(t), y''(t), z''(t) \rangle.$$

Example:

$$\vec{r}(t) = (x_0 + at, y_0 + bt, z_0 + ct)$$

$$\vec{v}(t) = \langle a, b, c \rangle \quad \text{CONSTANT.}$$

$$\vec{a}(t) = \langle 0, 0, 0 \rangle. \quad \text{NO ACCELERATION.}$$

Conversely, let $\vec{r}(t)$ be any path with zero acceleration:

$$\vec{r}''(t) = \langle 0, 0, 0 \rangle.$$

Integrate to find the velocity.

$$\vec{r}'(t) = \left\langle \int 0\,dt, \int 0\,dt, \int 0\,dt \right\rangle$$

$$= \langle a, b, c \rangle$$

for some constants of integration a, b, c. Then integrate velocity to get position:

$$\vec{r}(t) = \left\langle \int a\,dt, \int b\,dt, \int c\,dt \right\rangle$$

$$= \langle at + x_0, bt + y_0, ct + z_0 \rangle$$

for some constants of integration; call them x_0, y_0, z_0.

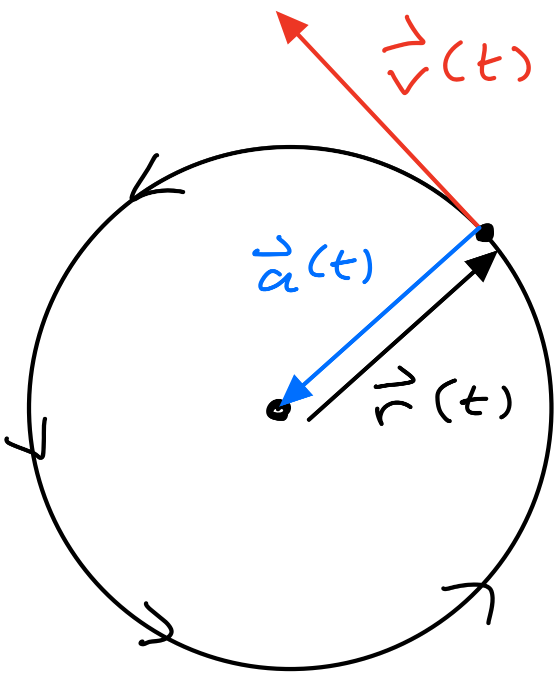
Conclusion: Any path with zero acceleration is a <u>straight line</u> with constant velocity.

Circle : $\vec{r}(t) = (\cos t, \sin t)$

$\vec{v}(t) = \langle -\sin t, \cos t \rangle$

$\vec{a}(t) = \langle -\cos t, -\sin t \rangle$.

Picture :



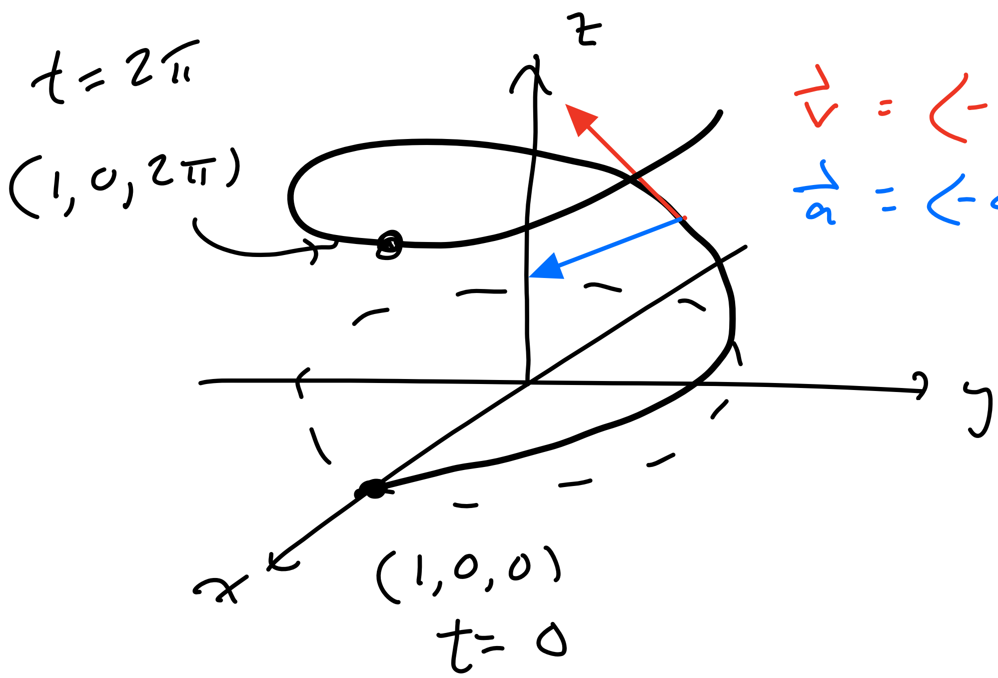
Acceleration always points directly to the center because.

$$\vec{a}(t) = \langle -\cos t, -\sin t \rangle = -\vec{r}(t).$$

Example in $\mathbb{R}^3$: Consider the following "helical" path

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$= (\cos t, \sin t, t)$$



$$\vec{v} = \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{a} = \langle -\cos t, -\sin t, 0 \rangle$$

Find arc length between $t = 0$ & $t = 2\pi$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{v}(t)\| = \sqrt{\underbrace{\cancel{\sin^2 t + \cos^2 t}}_{1} + 1}$$

$$= \sqrt{2}.$$

$$\text{Arc length} = \int_0^{2\pi} \|\vec{v}(t)\| \, dt$$

$$= \int_0^{2\pi} \sqrt{2} \, dt$$

$$= 2\pi\sqrt{2} .$$

Acceleration vector:

$$\vec{a}(t) = \vec{v}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$= \langle -\cos t, -\sin t, 0 \rangle .$$

Points directly toward z-axis.

Integration of vector-valued functions.

Given any function

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

we define the integral with respect to t "componentwise"

$$\int_{t_0}^{t_1} \vec{r}(t)\,dt =$$

$$\left\langle \int_{t_0}^{t_1} x(t)\,dt,\ \int_{t_0}^{t_1} y(t)\,dt,\ \int_{t_0}^{t_1} z(t)\,dt \right\rangle$$

Why do we want to do this?

Newton's 2nd Law.

If a time varying force $\vec{F}(t)$ acts on a particle of mass m at point $\vec{r}(t)$, Then we have

Force = mass · acceleration

$$\vec{F}(t) = m\,\vec{a}(t)$$
$$= m\,\vec{v}'(t)$$
$$= m r''(t).$$

This is a second order differential equation. Goal: Solve for $\vec{r}(t)$.

Example: Projectile Motion.

Near the surface of the Earth, a free falling body experiences a constant acceleration due to gravity:

$$\vec{r}''(t) = \langle 0, 0, -9.81\ m/s^2 \rangle$$

We can solve for $\vec{r}(t)$ by integrating twice.

$$\vec{r}'(t) = \left\langle \int 0\,dt, \int 0\,dt, \int -9.81\,dt \right\rangle$$

$$= \langle c_1, c_2, -9.81t + c_3 \rangle$$

Meaning of the constants:

$$\vec{r}'(0) = \langle \underbrace{c_1, c_2, 0 + c_3}_{\substack{\text{initial velocity.} \\ \text{Call it } u_0, v_0, w_0}} \rangle$$

$$\vec{r}'(t) = \langle u_0, v_0, -9.81t + w_0 \rangle.$$

Integrate again to get position at time t:

$$\vec{r}(t) = \left\langle \int \quad dt, \int \quad dt, \int (-9.81t + \quad)dt \right\rangle$$

$$= \left\langle u_0 t + d_1,\ v_0 t + d_2,\ -\frac{9.81}{2}t^2 + w_0 t + d_3 \right\rangle$$

Meaning of d_1, d_2, d_3:

$$\vec{r}(0) = \langle \underbrace{d_1, d_2, d_3}_{\text{initial position.}} \rangle$$

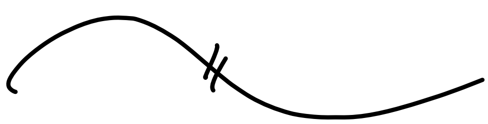
initial position.
call x_0, y_0, z_0

A free falling body has position

$$\vec{r}(t) = \langle \underbrace{x_0 + u_0 t,\ y_0 + v_0 t,\ z_0 + w_0 t}_{\text{this part is like a straight line}} - \frac{9.81}{2}t^2 \rangle$$

this part is like a straight line

this makes it curve toward Earth.



More interesting:

Universal Gravitation.

Put the sun at $(0,0,0)$ in $\mathbb{R}^3$.

A moving planet has position $\vec{r}(t)$.

Newton: Planet feels a gravitational force of magnitude

$$\frac{GMm}{\|\vec{r}(t)\|^2}$$

where M = mass of sun
m = mass of planet
G = gravitational constant.

The direction of the force is directly toward the sun.

SOLVE FOR $\vec{r}(t)$.