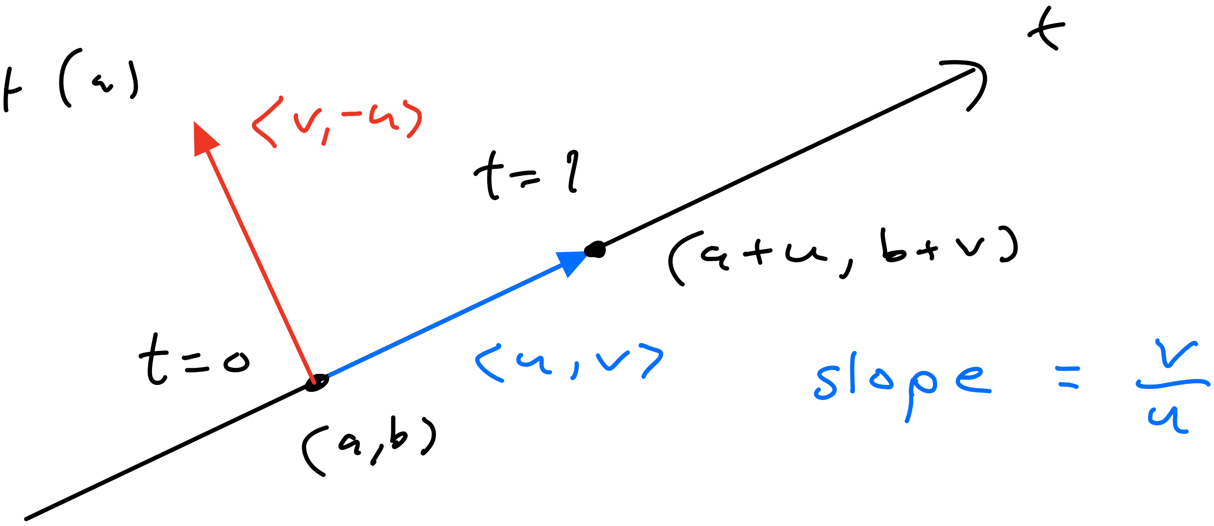
Today: HW 1 Discussion.

Monday: Quiz 1 at the beginning of class for 20 minutes (11:40AM – 12:00 PM)

Problem 1: Lines & Circles

Part (a)



This gives a parametrization

$$(x, y) = (a + ut, b + vt).$$

velocity

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (u, v)$$

CONSTANT VELOCITY!

Conversely: We will see later that any curve of constant velocity must be a straight line.

speed

$$\| \langle u, v \rangle \| = \sqrt{u^2 + v^2}$$

ALSO CONSTANT.

We can eliminate t:

$x = a + ut \longrightarrow t = (x-a)/u$

$y = b + vt \longrightarrow t = (y-b)/v$

$$\frac{(x-a)}{u} = \frac{(y-b)}{v} \quad \text{if } u \neq 0,\ v \neq 0$$

$(x-a)v = (y-b)u$ now $u = 0$ or $v = 0$ OK ✓

$vx - uy = av - bu$

The equation of a line that is $\perp$ to vector $\langle v, -u \rangle$

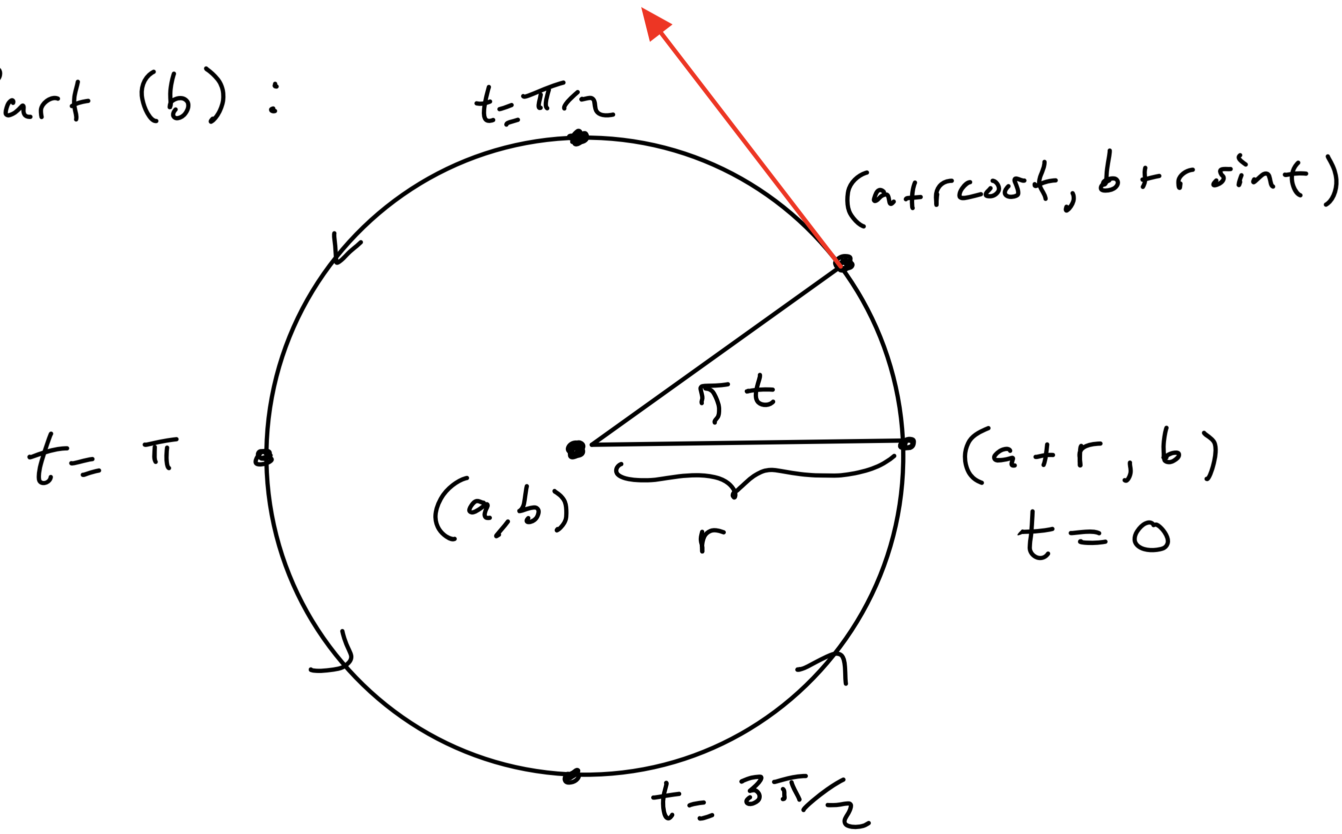
Slope - Intercept :

$$uy = vx - av + bu$$

$$y = \frac{v}{u}x + b - \frac{av}{u}.$$

slope

Part (b) :



$$(x,y) = (a + r\cos t, b + r\sin t)$$

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (-r\sin t, r\cos t)$$

NOT CONSTANT.

$$\text{speed} = \sqrt{(-r\sin t)^2 + (r\cos t)^2}$$

$$= \sqrt{r^2\sin^2 t + r^2\cos^2 t}$$

$$= \sqrt{r^2 \left(\cancel{\sin^2 t + \cos^2 t}\right)}$$

$$= \sqrt{r^2} = |r| = r.$$

CONSTANT SPEED :)

We can eliminate t:

Use $\sin^2 t + \cos^2 t = 1$.

$$x = a + r\cos t \rightsquigarrow \cos t = (x-a)/r$$

$$y = b + r\sin t \rightsquigarrow \sin t = (y-b)/r$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x-a}{r}\right)^2 + \left(\frac{y-b}{r}\right)^2 = 1.$$

$$(x-a)^2 + (y-b)^2 = r^2$$

Circle centred at (a,b) with radius r.

Problem 2:

$$(x, y) = (t^2 - 1, t^3 - t)$$

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (2t, 3t^2 - 1)$$

slope of the tangent line at time t:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{2t}$$

When is it $0, \pm 1, \infty$ (vertical)

$t = 0 \rightsquigarrow$ vertical tangent

$$(x, y) = (0^2 - 1, 0^3 - 0) = (-1, 0)$$

Slope 0: $\quad \dfrac{3t^2 - 1}{2t} = 0 \quad (t \neq 0)$

$$3t^2 - 1 = 0 \longrightarrow t = \pm\sqrt{1/3}$$

points:

$$(x, y) = \left(\frac{1}{3} - 1, \left(\sqrt{\frac{1}{3}}\right)^3 - \sqrt{\frac{1}{3}}\right) \qquad t = \sqrt{\frac{1}{3}}$$

$$= (-0.67, 0.3)$$

$$(x,y) = \left(\frac{1}{3}-1, \left(-\sqrt{\frac{1}{3}}\right)^3 + \sqrt{\frac{1}{3}}\right)$$
$$= (-0.67, -0.3) \qquad t = -\sqrt{\frac{1}{3}}$$

Slope ± 1:

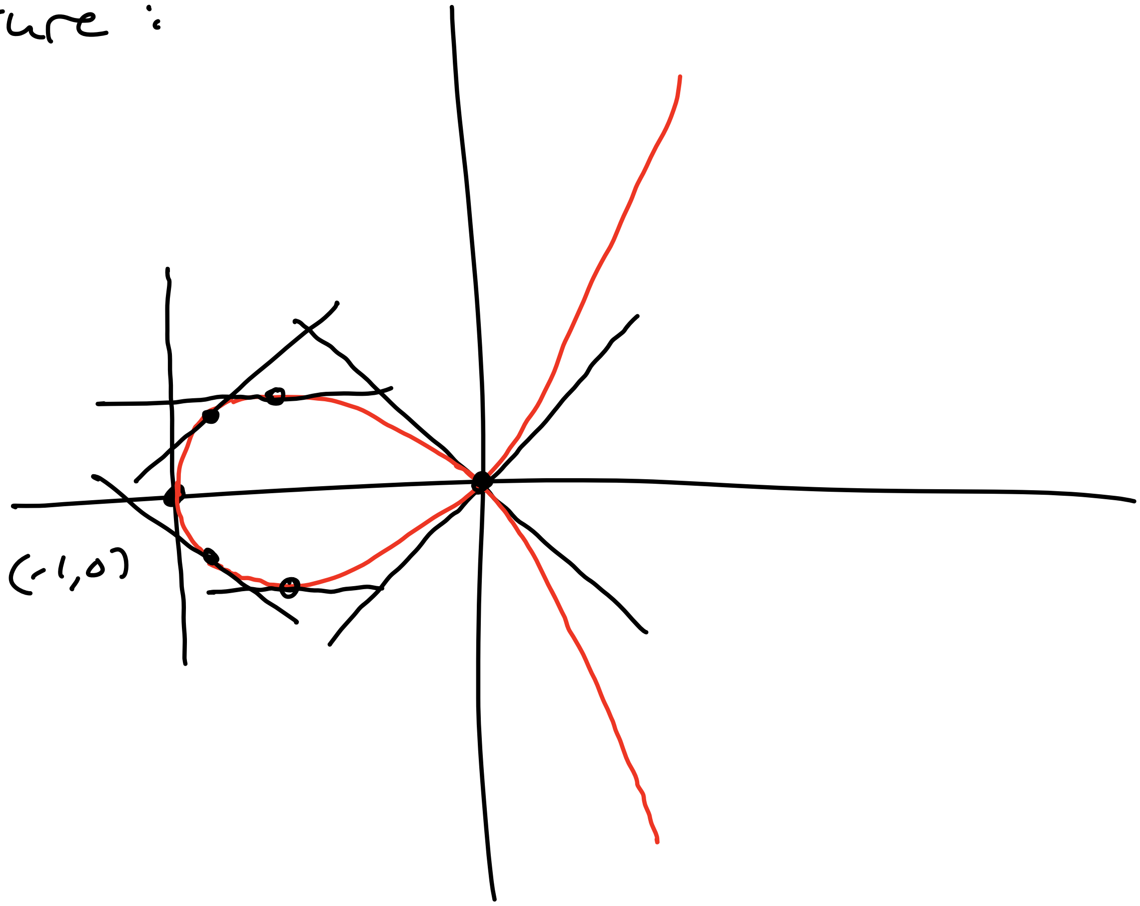
$$\frac{3t^2-1}{2t} = +1 \rightarrow 3t^2-1 = 2t$$
$$3t^2-2t-1=0$$
$$t = \frac{2 \pm \sqrt{4+12}}{6}$$
$$= \frac{2\pm 4}{6} = 1 \text{ or } -\frac{1}{3}$$

$(0,0)$ $(-.87, 0.3)$

$$\frac{3t^2-1}{2t} = -1 \rightsquigarrow 3t^2-1 = -2t$$
$$3t^2+2t-1 = 0$$
$$t = -1 \text{ or } +\frac{1}{3}$$

$(0,0)$ $(-0.87, -0.3)$

Picture:



Problem 3: The Cycloid.

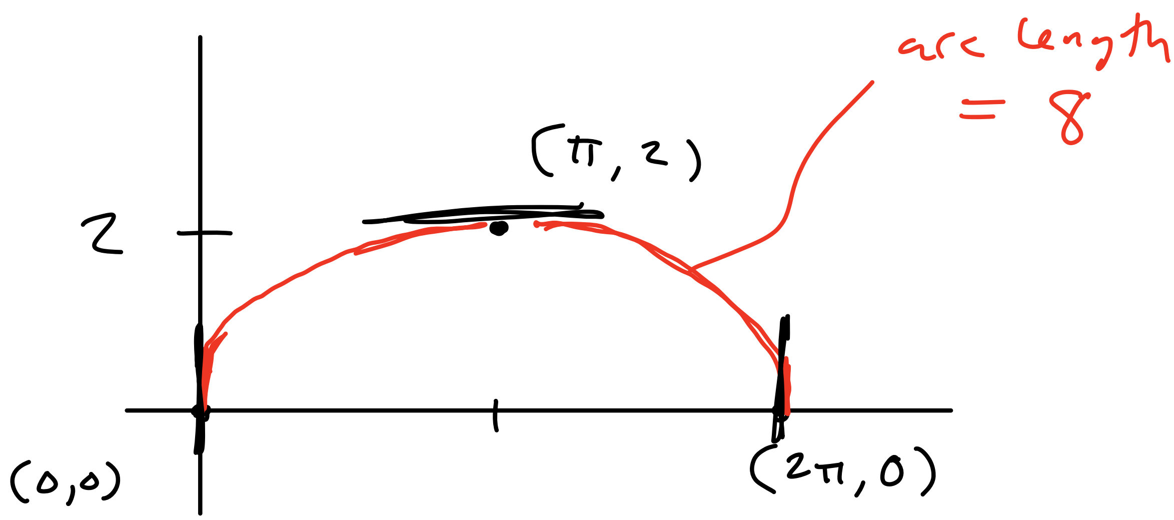
$$(x,y) = (t - \sin t,\ 1 - \cos t)$$

sketch between $t=0$, $t=2\pi$.

$t=0$: $(0 - \sin 0,\ 1 - \cos 0) = (0,0)$

$t=2\pi$: $(2\pi - \underset{0}{\cancel{\sin 2\pi}},\ 1 - \underset{1}{\cancel{\cos 2\pi}}) = (2\pi, 0)$

$t=\pi$: $(\pi - \underset{0}{\cancel{\sin \pi}},\ 1 - \underset{-1}{\cancel{\cos \pi}}) = (\pi, 2)$



slope of tangent $\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$.

$t = \pi$: slope 0

$t = 0$ or $t = 2\pi$: slope $\to \infty$.

As $t \to 0$, $\frac{\sin t}{1-\cos t} \to \frac{0}{0}$ (oops!)

L'Hôpital's Rule

$$\lim_{t\to 0} \frac{\sin t}{1-\cos t} = \lim_{t \to 0} \frac{\cos t}{\sin t} = \infty$$

Arc length:

velocity $= \langle 1-\cos t, \sin t \rangle$

$$\text{speed}^2 = (1-\cos t)^2 + \sin^2 t$$

[$\sin^2 t = (\sin t)^2$. weird ...]

$$= 1 - 2\cos t + \underbrace{\cancel{\cos^2 t + \sin^2 t}}_{1}$$

$$= 2 - 2\cos t$$

$$= 2(1-\cos t)$$

$$\text{speed} = \sqrt{2(1-\cos t)}$$

Something Lucky:

Half-Angle Formula

$$\sin\left(\frac{t}{2}\right) = \sqrt{\frac{1-\cos t}{2}}$$

$$\sin^2\left(\frac{t}{2}\right) = \frac{1-\cos t}{2}$$

$$1-\cos t = 2\sin^2\left(\frac{t}{2}\right)$$

$$2(1-\cos t) = 4\sin^2\left(\frac{t}{2}\right).$$

so . . .

$$\text{speed} = \sqrt{2(1-\cos t)}$$

$$= \sqrt{4\sin^2(t/2)}$$

$$= 2\sin(t/2)$$

Finally: Arc length

$$= \int_0^{2\pi} \text{speed}\, dt$$

$$= \int_{t=0}^{t=2\pi} 2\sin\left(\frac{t}{2}\right) dt$$

$u = t/2$
$du = dt/2$
$dt = 2du.$

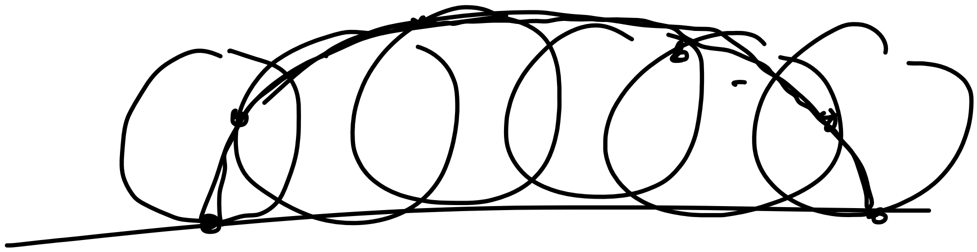
$$= \int_{u=0}^{u=\pi} 2\sin(u)\cdot 2du.$$

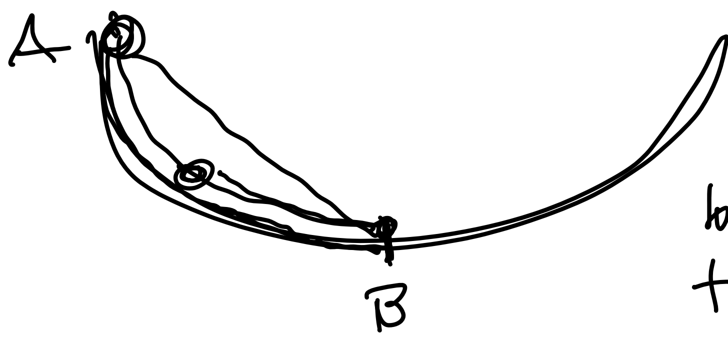
$$= 4\left[-\cos(u)\right]_0^{\pi}$$

$$= 4\left[\underbrace{-\cos(\pi)}_{1} + \underbrace{\cos(0)}_{1}\right]$$

$$= 8$$ weird !

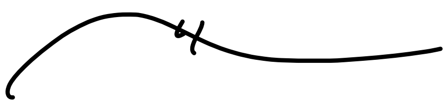
Physics :



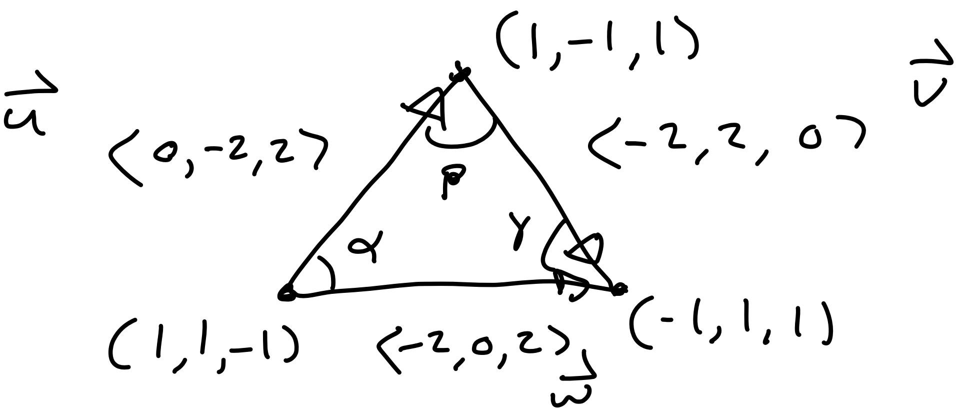


brachistochrone

tautochrone



Problem 4 : Triangle in Space.



$$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| =$$

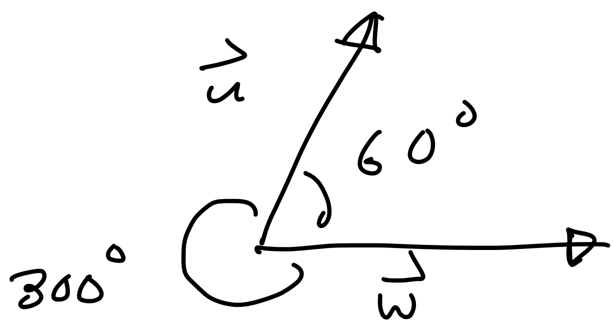
$$\sqrt{0^2+(-2)^2+2^2} = \sqrt{8} = 2\sqrt{2}.$$

$$\cos\alpha = \frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\,\|\vec{w}\|} = \frac{(0)(-2)+(-2)(0)+(2)(2)}{\sqrt{8}\cdot\sqrt{8}}$$

$$= \frac{4}{8} = \frac{1}{2}$$

$$\alpha = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \text{ or } 300^\circ$$

pick the smaller.

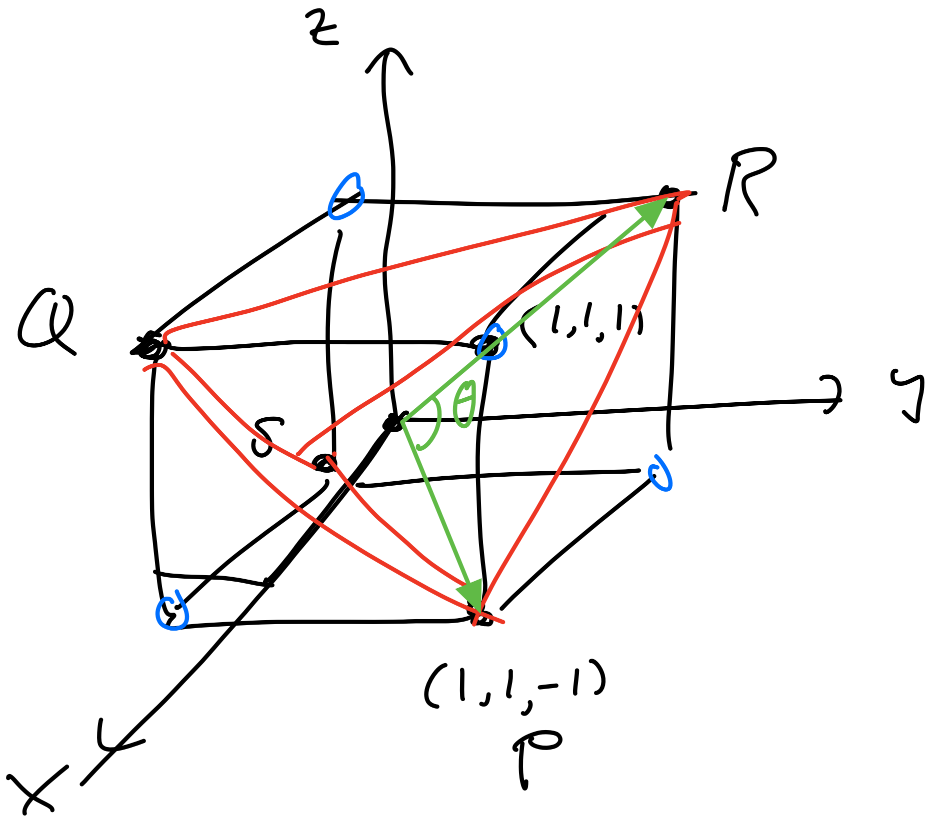


Remark: Add 4th point $S = (-1,-1,-1)$

Then PQR, PQS, PRS, QRS

all equilateral triangles

PQRS is a regular tetrahedron



Call the origin $O = (0,0,0)$

The tetrahedral angle θ between any two vertices, measured from the central point O.

θ = angle between $\overrightarrow{OP}$ & $\overrightarrow{OR}$

$$\cos\theta = \frac{\overrightarrow{OP} \cdot \overrightarrow{OR}}{\|\overrightarrow{OP}\| \, \|\overrightarrow{OR}\|}$$

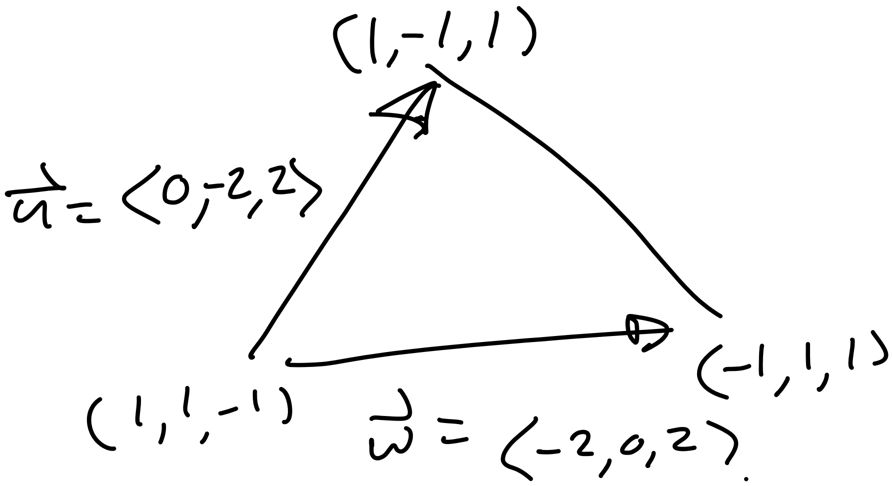
$$= \frac{\langle 1,1,-1\rangle \cdot \langle -1,1,1\rangle}{\|\langle 1,1,-1\rangle\| \, \|\langle -1,1,1\rangle\|}$$

$$= \frac{-1+1-1}{\sqrt{3}\cdot\sqrt{3}} = -\frac{1}{3}$$

$$\theta = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109.5^\circ$$

Problem 6: Too easy!

Here's a harder version. Find equation of the plane containing points P, Q, R from Problem 4.



Need one point ✓ & the normal vector.

To get a normal vector, take cross product of any two vectors in the plane. e.g. $\vec{u} = \langle -2, 2, 0 \rangle$

$\vec{w} = \langle -2, 0, 2 \rangle$.

$$\vec{u} \times \vec{w} = \text{“} \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix} \text{”}$$

$$= \vec{i} \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$- \vec{j} \det \begin{pmatrix} -2 & 0 \\ -2 & 2 \end{pmatrix}$$

$$+ \vec{k} \det \begin{pmatrix} -2 & 2 \\ -2 & 0 \end{pmatrix}$$

$$= (4-0)\vec{i} - (-4-0)\vec{j} + 4\vec{k}$$

$$= 4\vec{i} + 4\vec{j} + 4\vec{k}$$

$$= \langle 4, 4, 4 \rangle$$

Using normal vector $\langle 4,4,4 \rangle$ and any point, say $P = (1,1,-1)$, the equation of the plane is

$$4(x-1) + 4(y-1) + 4(z-(-1)) = 0$$

$$4x + 4y + 4z = 4$$

$$x + y + z = 1$$

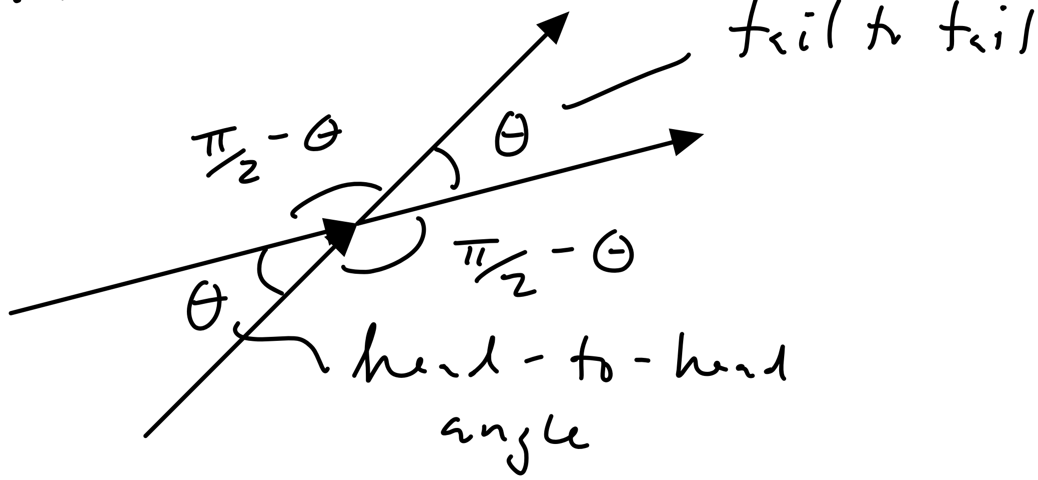
In retrospect this was very easy to see. Indeed the x, y, z coordinates of any point in the plane sum to 1.

$P = (1,1,-1) \longrightarrow 1+1-1 = 1$

$Q = (1,-1,1) \longrightarrow 1-1+1 = 1$

$R = (-1,1,1) \longrightarrow -1+1+1 = 1$

Remark :



tail to tail
θ
$\pi/2 - \theta$
$\pi/2 - \theta$
θ
head-to-head
angle