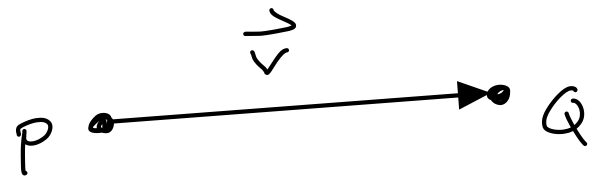
HW1 is due Fri before class, on Blackboard.

A vector is an ordered pair of points in the plane (or in space of any # of dimensions). We view it as an arrow:



"$\vec{v} = \overrightarrow{PQ}$"

Coordinates: If $P = (p_1, p_2, p_3)$
$Q = (q_1, q_2, q_3)$

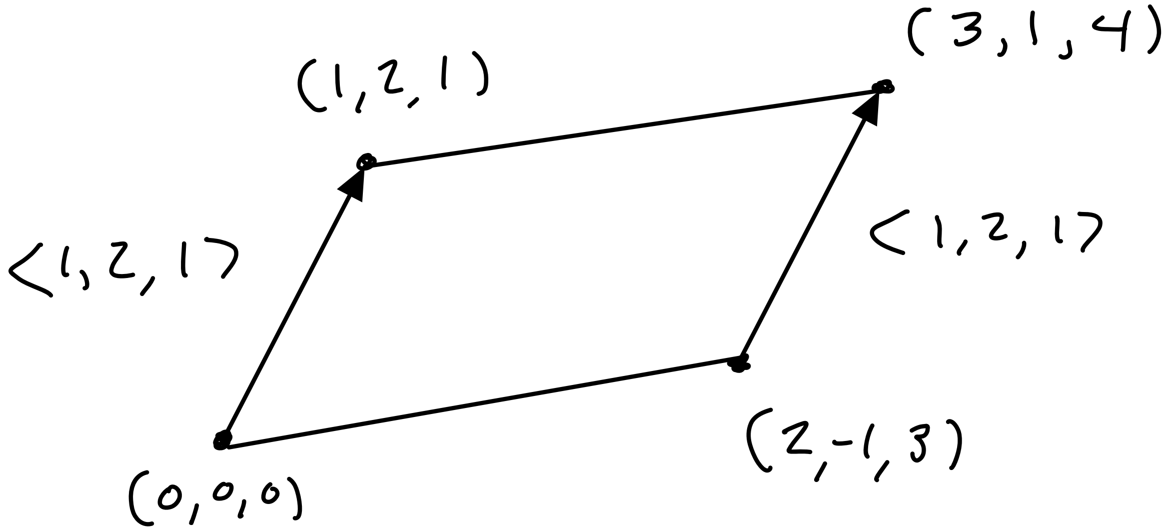
then write

$$\vec{v} = \overrightarrow{PQ} = \text{"head minus tail"}$$

$$= \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$

Vectors can be moved around

without changing their coordinates:



Vectors with tail at (0,0,0) are in "standard position".

Sometimes we discuss vectors without mentioning the endpoints.

There are 3 basic operations:

Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

define

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

$$k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \quad \text{(strange)}$$

These operations satisfy quite a few "obvious rules".

[Remark: we have

vector + vector = vector

scalar · vector = vector

vector • vector = scalar

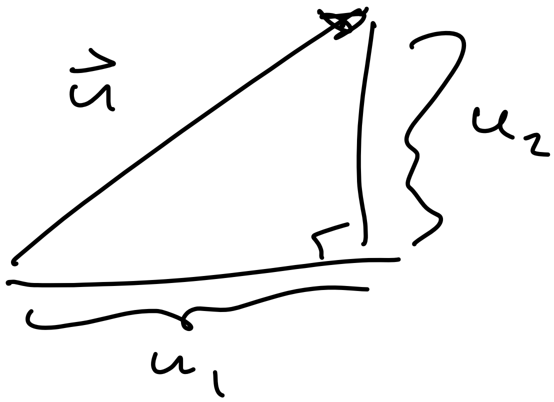
There is no general way to "multiply" vectors to get a vector:

vector × vector = vector ?

However, in 3D there is a strange operation called "cross product".]

Length (or magnitude) of a vector is computed using the Pythagorean theorem:

$$\vec{u} = \langle u_1, u_2 \rangle$$

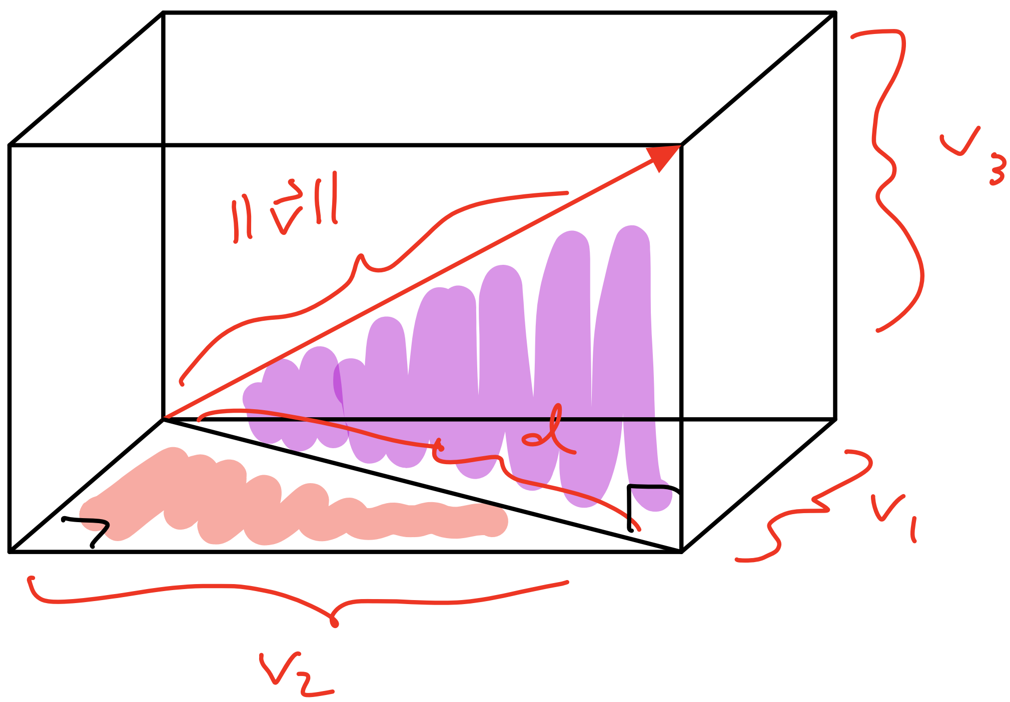


Let $\|\vec{u}\|$ be the length of $\vec{u}$.

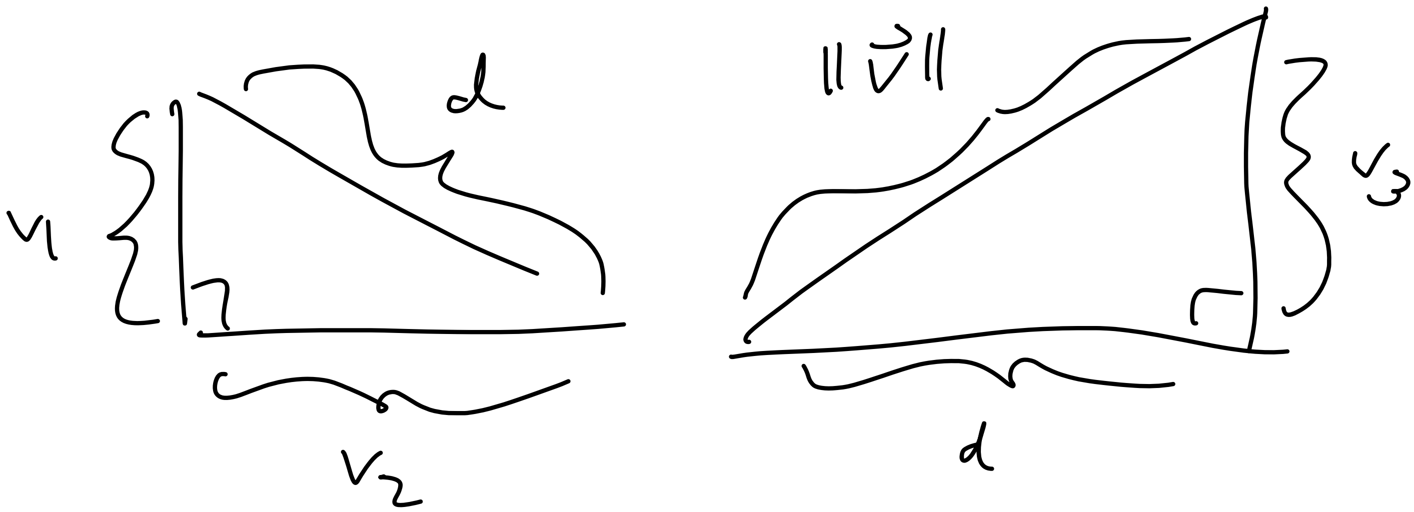
Then $\|\vec{u}\|^2 = u_1^2 + u_2^2$

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$$

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$.



There are 2 right triangles here:



$$d^2 = v_1^2 + v_2^2 \qquad \|\vec{v}\|^2 = d^2 + v_3^2$$

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

[By analogy: We use the same definition in any # of dimensions.

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$]

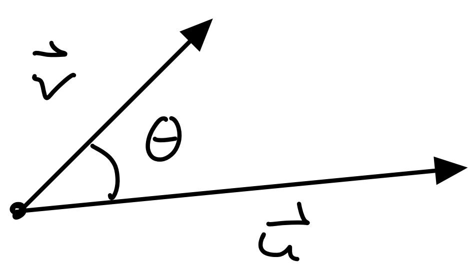
Dot Product is related to Lengths & Angles :

- Length :

$$\vec{v} \cdot \vec{v} = \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$$
$$= v_1 v_1 + v_2 v_2 + v_3 v_3$$
$$= v_1^2 + v_2^2 + v_3^2$$
$$= \| \vec{v} \|^2$$

$$\boxed{\vec{v} \cdot \vec{v} = \| \vec{v} \|^2}$$

- Angles : Let θ be the angle between vectors $\vec{u}$ & $\vec{v}$ placed "tail-to-tail"



Dot Product Theorem:

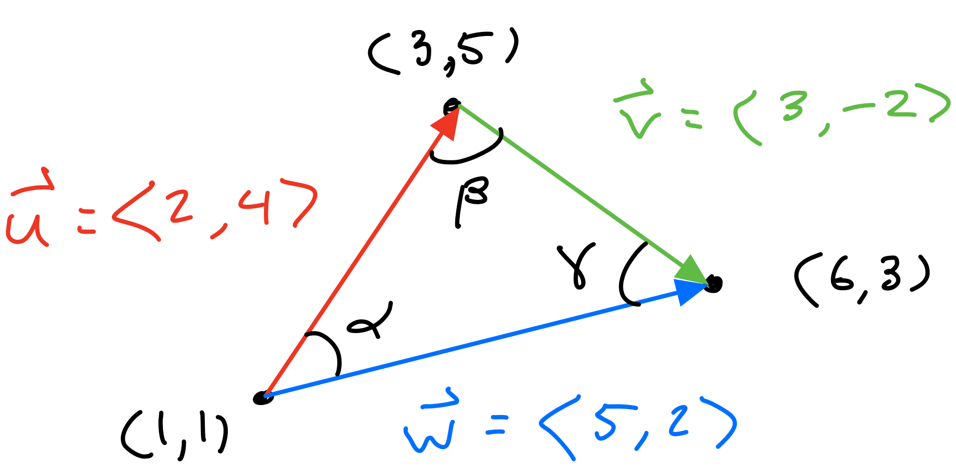
$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$

Combine these boxed formulas:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\sqrt{\vec{u} \cdot \vec{u}} \cdot \sqrt{\vec{v} \cdot \vec{v}}}$$

Example: Consider the triangle.



Compute the angles α, β, γ.

First compute the dot products:

$$\vec{u} \cdot \vec{u} = 2^2 + 4^2 = 20 \rightsquigarrow \|\vec{u}\| = \sqrt{20}$$

$$\vec{v} \cdot \vec{v} = 3^2 + (-2)^2 = 13 \longrightarrow \|\vec{v}\| = \sqrt{13}$$

$$\vec{w} \cdot \vec{w} = 5^2 + 2^2 = 29 \rightsquigarrow \|\vec{w}\| = \sqrt{29}$$

$$\vec{u} \cdot \vec{v} = 2 \cdot 3 + 4 \cdot (-2) = -2$$

$$\vec{u} \cdot \vec{w} = 2 \cdot 5 + 4 \cdot 2 = 18$$

$$\vec{v} \cdot \vec{w} = 5 \cdot 3 + 2 \cdot (-2) = 11.$$

Compute α:

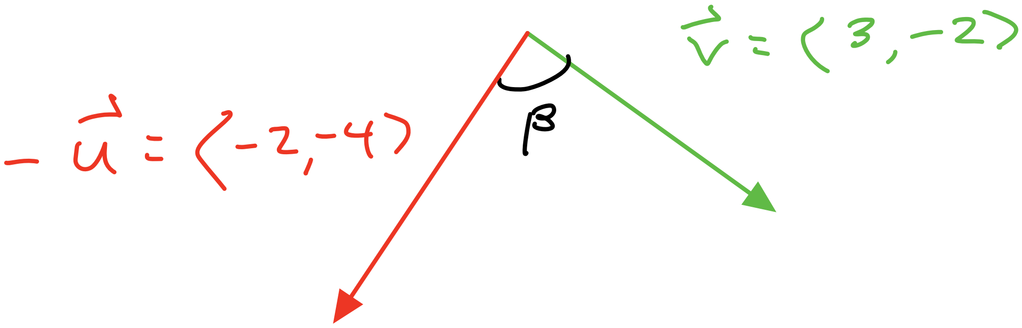
Since $\vec{u}$ & $\vec{w}$ are tail-to-tail:

$$\cos \alpha = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \, \|\vec{w}\|}$$

$$= \frac{18}{\sqrt{20} \cdot \sqrt{29}} \rightsquigarrow \alpha = 41.68^\circ$$

Compute β: Since $\vec{u}$ & $\vec{v}$ are not tail-to-tail,

Use $-\vec{u}$ instead of $\vec{u}$:



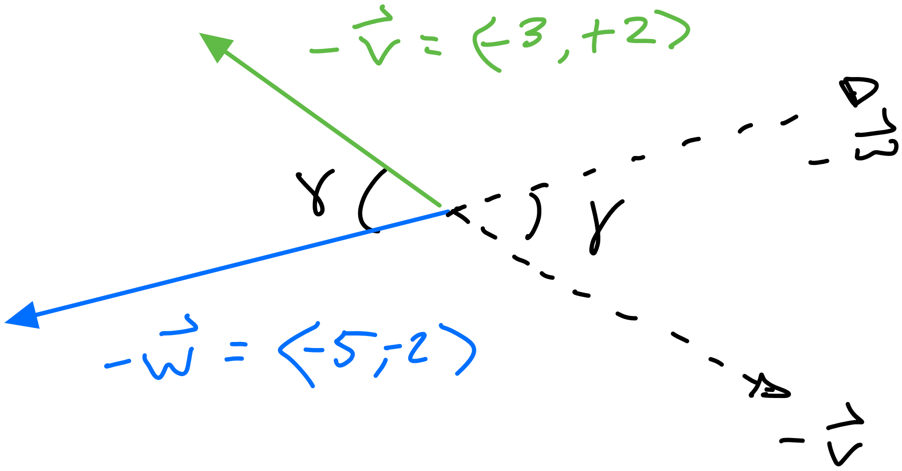
Observe: $\|-\vec{u}\| = \|\vec{u}\| = \sqrt{20}$

$$(-\vec{u}) \cdot \vec{v} = -(\vec{u} \cdot \vec{v}) = +2$$

$$\cos\beta = \frac{(-\vec{u}) \cdot \vec{v}}{\|-\vec{u}\| \cdot \|\vec{v}\|} = \frac{2}{\sqrt{20} \cdot \sqrt{13}}$$

$$\longrightarrow \quad \beta = 82.85^\circ$$

Compute γ: Note $-\vec{v}$ & $-\vec{w}$ are tail to tail:



$$\cos\gamma = \frac{(-\vec{v})\cdot(-\vec{w})}{\|-\vec{v}\|\cdot\|-\vec{w}\|}$$

$$= \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$$

$$\left[\ (-\vec{v})\circ(-\vec{w}) = -\left(\vec{v}\circ(-\vec{w})\right) = \vec{v}\circ\vec{w}\ \right]$$

$$= \frac{11}{\sqrt{13}\,\sqrt{29}}$$

$$\rightsquigarrow \quad \gamma = 55.49^\circ$$

Check:

$$\alpha + \beta + \gamma = 41.63^\circ + 88.85^\circ + 55.49^\circ$$

$$= 180^\circ \quad \checkmark$$

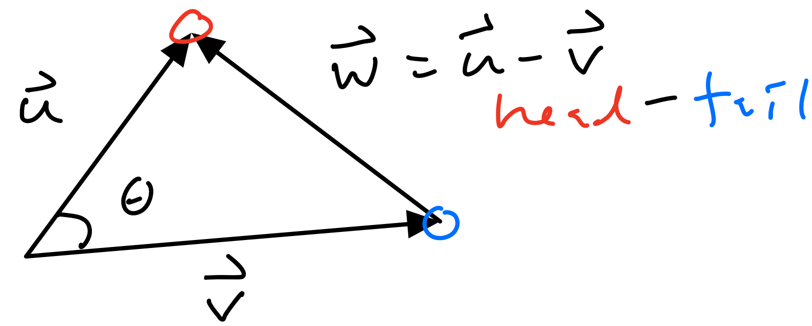
See HW 1 Problem 4 for an example in 3D.

HW 1 Problem 5 is about the

Dot Product Theorem:

Let θ be angle between vectors

$\vec{u}$ & $\vec{v}$ placed tail to tail:



Let $\vec{w} = \vec{u} - \vec{v}$

$\vec{u} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$

Geometry:

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Algebra:

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$$

Don't think!

$$= \vec{u} \cdot (\vec{u} - \vec{v}) - \vec{v} \cdot (\vec{u} - \vec{v})$$

[Compare $(a+b)c = ac + bc$]

$$= \cancel{\vec{u}\cdot\vec{u}} - \vec{u}\cdot\vec{v} - (\vec{v}\cdot\vec{u} - \cancel{\vec{v}\cdot\vec{v}})$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u}$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\,\vec{u}\cdot\vec{v}.$$

[use $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$]

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v}).$$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v})$$

Now you can think again.

same.

Compare to the geometry:

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

ALGEBRA $\longleftrightarrow$ GEOMETRY

Equations of Lines & Planes.

The gradeschool equation of a line is

$$y = mx + b.$$

But this equation does not generalize to higher dimensions.

In this class we prefer to write

$$ax + by = c.$$

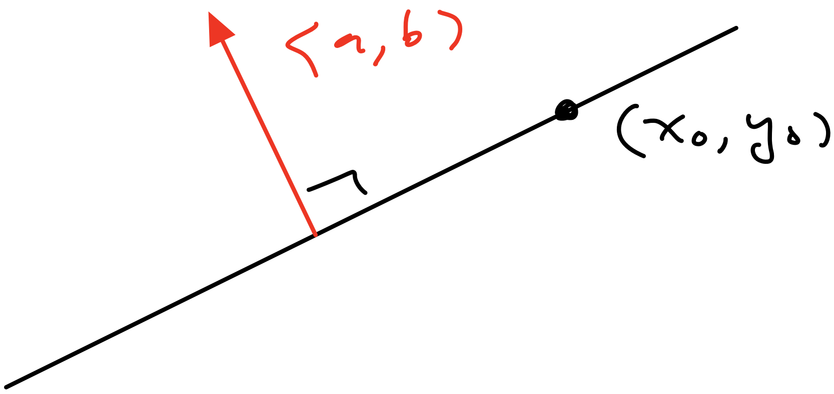
Even better:

$$\boxed{a(x-x_0) + b(y-y_0) = 0}$$

What does it mean?

I claim this is the line that

- passes through point (x_0, y_0)
- is perpendicular to vector $\langle a,b \rangle$.



Key: Dot product Theorem

$$\vec{u} \circ \vec{v} = \underbrace{\|\vec{u}\| \cdot \|\vec{v}\|}_{\neq 0} \cos\theta$$

Note that

$$\vec{u} \circ \vec{v} = 0 \iff \cos\theta = 0$$

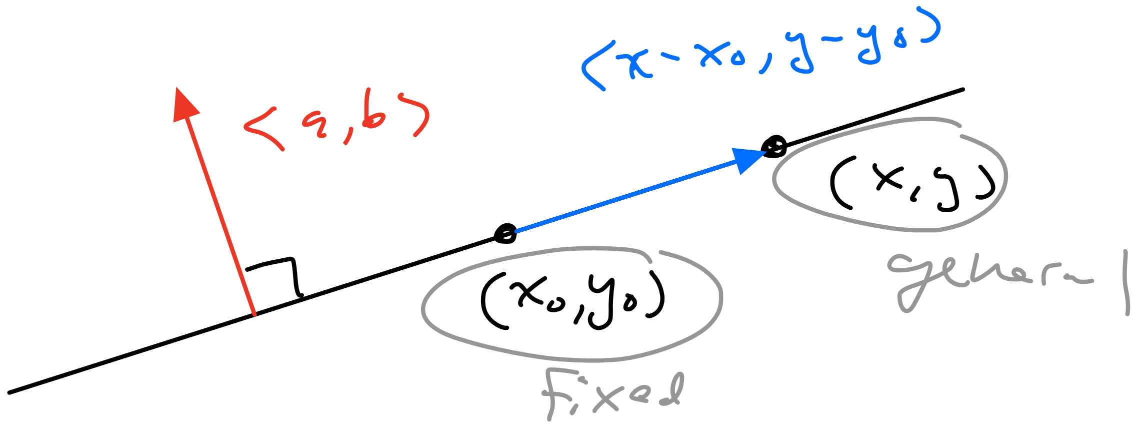
$$\iff \vec{u} \perp \vec{v}$$

"perpendicular"

$$\boxed{\vec{u} \circ \vec{v} = 0 \iff \vec{u} \perp \vec{v}}$$

So let L be the line passing through (x_0, y_0) and $\perp$ to the vector $\langle a, b \rangle$.

Then for any general point (x, y) on the line we have



Since the vector $\langle x-x_0, y-y_0 \rangle$ is parallel to L & $\langle a, b \rangle$ is $\perp$ to L we get

$$\langle a, b \rangle \perp \langle x-x_0, y-y_0 \rangle$$

hence

$$\langle a, b \rangle \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$a(x-x_0) + b(y-y_0) = 0 \quad \checkmark$$

We can convert into $y = mx + b$ form if we want.

Example: Consider the line in $\mathbb{R}^2$ passing through point $(1,2)$ & perp. to vector $\langle 3,1 \rangle$.

Let $(x_0, y_0) = (1,2)$

$\langle a, b \rangle = \langle 3, 1 \rangle$

Equation of the line:

$$a(x-x_0) + b(y-y_0) = 0$$

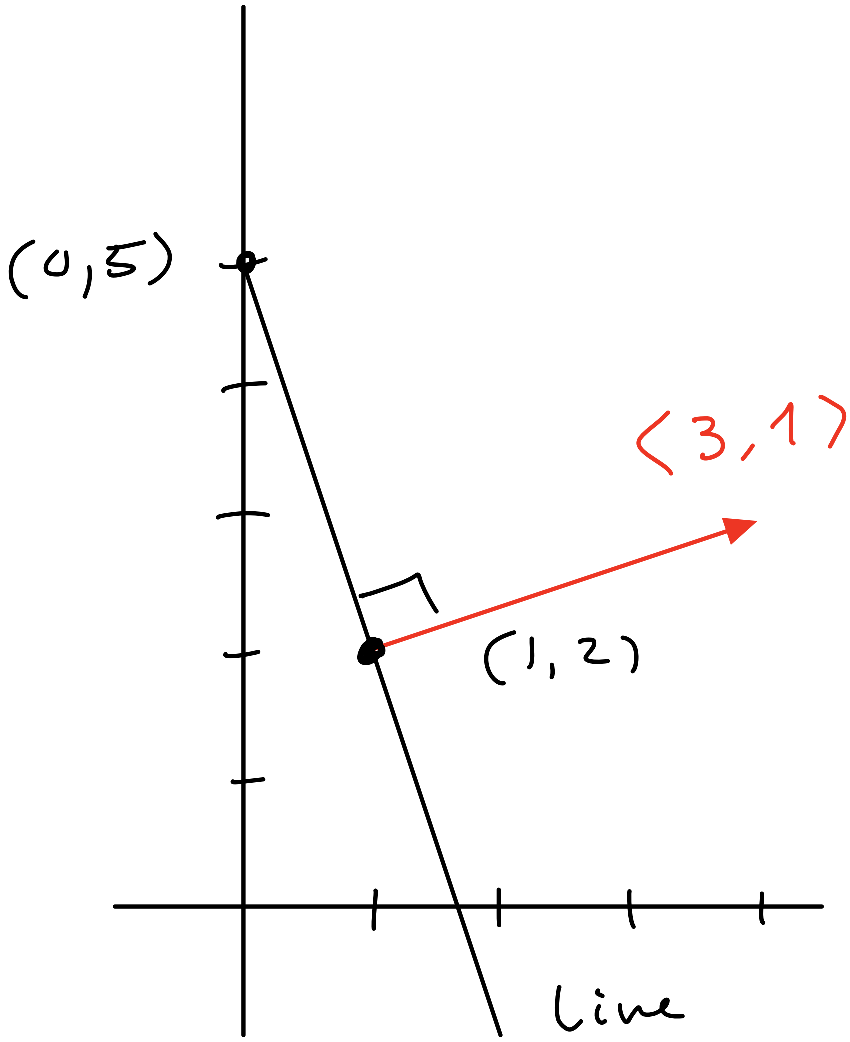
$$3(x-1) + 1(y-2) = 0$$

$$3x - 3 + y - 2 = 0$$

$$3x + y - 5 = 0$$

$$y = \underbrace{-3}_{\text{slope}}x + \underbrace{5}_{\text{y-intercept}}$$

Picture:



line

$$3(x-1)+1(y-2)=0$$

or

$$y=-3x+5$$

Note: The form of the equation is not unique. We could pick any point on the line

& any vector $\perp$ to line.

e.g. $(x_0, y_0) = (0, 5)$

$\langle a, b \rangle = \langle 6, 2 \rangle$

Check:

$$a(x - x_0) + b(y - y_0) = 0$$

$$6(x - 0) + 2(y - 5) = 0$$

$$6x - 0 + 2y - 10 = 0$$

$$6x + 2y - 10 = 0$$

divide by 2

$$3x + y - 5 = 0$$

$$y = -3x + 5$$

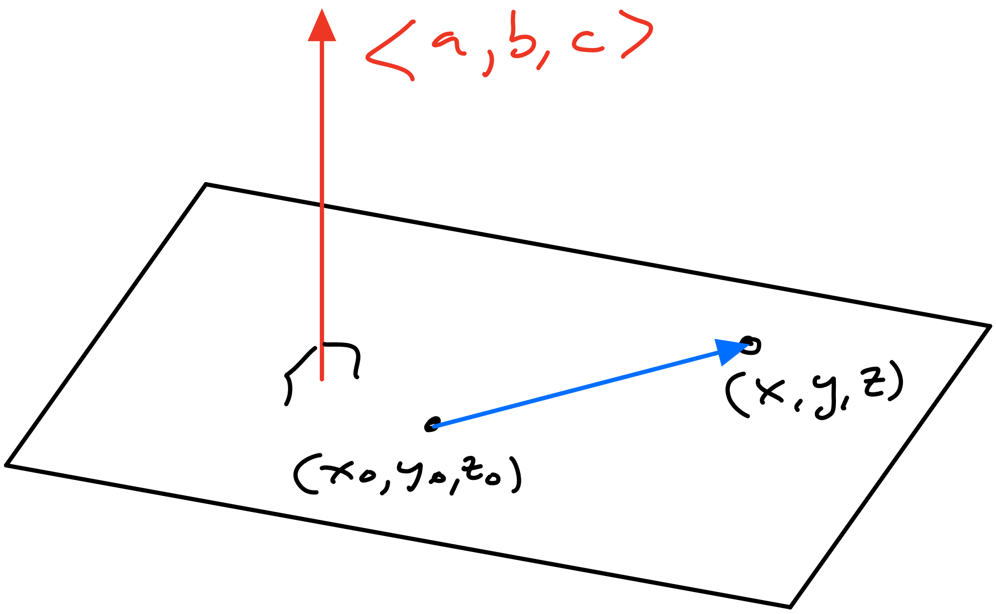
SAME ✓

Question: What shape is represented by following eq?

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This is a plane in 3D !

Picture :



The plane contains (x_0, y_0, z_0) and is $\perp$ to vector $\langle a, b, c \rangle$ so for any point (x, y, z) in plane we have

$$\langle a, b, c \rangle \perp \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$\underbrace{a(x - x_0) + b(y - y_0) + c(z - z_0)}_{\text{dot product.}} = 0$$