This Week: Chapter 5

The mathematical core of Calculus is differentiation and integration of functions. Here is a summary:

Kind of function	diff	int
$\mathbb{R} \to \mathbb{R}$	Calc I	Calc I & II
$\mathbb{R} \to \mathbb{R}^n$	Ch 1 & 3	Ch 3 & 6
$\mathbb{R}^n \to \mathbb{R}$	Ch 4	Ch 5
$\mathbb{R}^m \to \mathbb{R}^n$	Ch 6	Ch 6

DONE.

NOW.

We can't cover this completely.

Chapter 5: Integration of scalar fields in $\mathbb{R}^2$ & $\mathbb{R}^3$.

Example: Consider $f(x,y) = xy^2$.

Think of this as the height of a surface above x,y-plane:

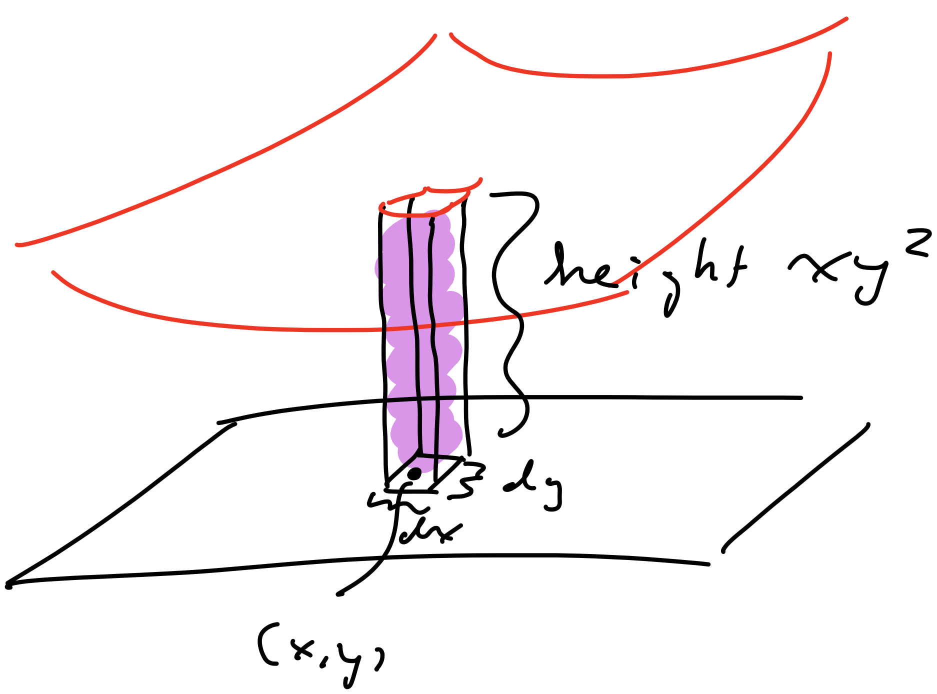
[see Geogebra].

Compute the volume of solid region above the square

$$0 \le x \le 1$$
$$0 \le y \le 1$$

and below the surface.

Idea: Consider a skinny vertical column above the point (x,y):



volume of skinny column

$= \text{height} \times \text{area of base}$

$= xy^2\,dx\,dy.$

To obtain volume of the region, "add up all the skinny columns":

$$\text{volume} = \iint xy^2\,dx\,dy$$

how to compute?

It's just two integrals, one with respect to x & one with respect to y, and the <u>order doesn't matter</u>. Let's integrate x first.

$$\text{vol} = \int_{y=0}^{y=1} \left(\int_{x=0}^{x=1} xy^2\,dx \right) dy$$

$$= \int_{y=0}^{y=1} \left[\frac{1}{2} x^2 y^2 \right]_0^1 dx$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{2} y^2 - 0 \right) dy$$

$$= \int_0^1 \frac{1}{2} y^2 \, dy$$

$$= \left[\frac{1}{2} \cdot \frac{1}{3} y^3 \right]_0^1$$

$$= \frac{1}{6} - 0 = 1/6$$

The exact volume of the 3D region is $1/6$.

Check that order doesn't matter:

Now do y first:

$$\text{vol} = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} x y^2 \, dy \right) dx$$

$$= \int_{x=0}^{x=1} \left[\frac{1}{3} x y^3 \right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=1} \left(\frac{1}{3} x - 0 \right) dx$$

$$= \int_0^1 \frac{1}{3} x \, dx$$

$$= \left[\frac{1}{3} \cdot \frac{1}{2} x^2 \right]_{x=0}^{x=1}$$

$$= \frac{1}{6} - 0 = 1/6 \checkmark$$

Integrate over a different region:

$$\int_{y=0}^{y=1} \left(\int_{x=-1}^{x=1} x y^2 \, dx \right) dy$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{2} x^2 y^2 \right)_{x=-1}^{x=1} dy$$

$$= \int_0^1 \left(\frac{1}{2}y^2 - \frac{1}{2}y^2\right) dy$$

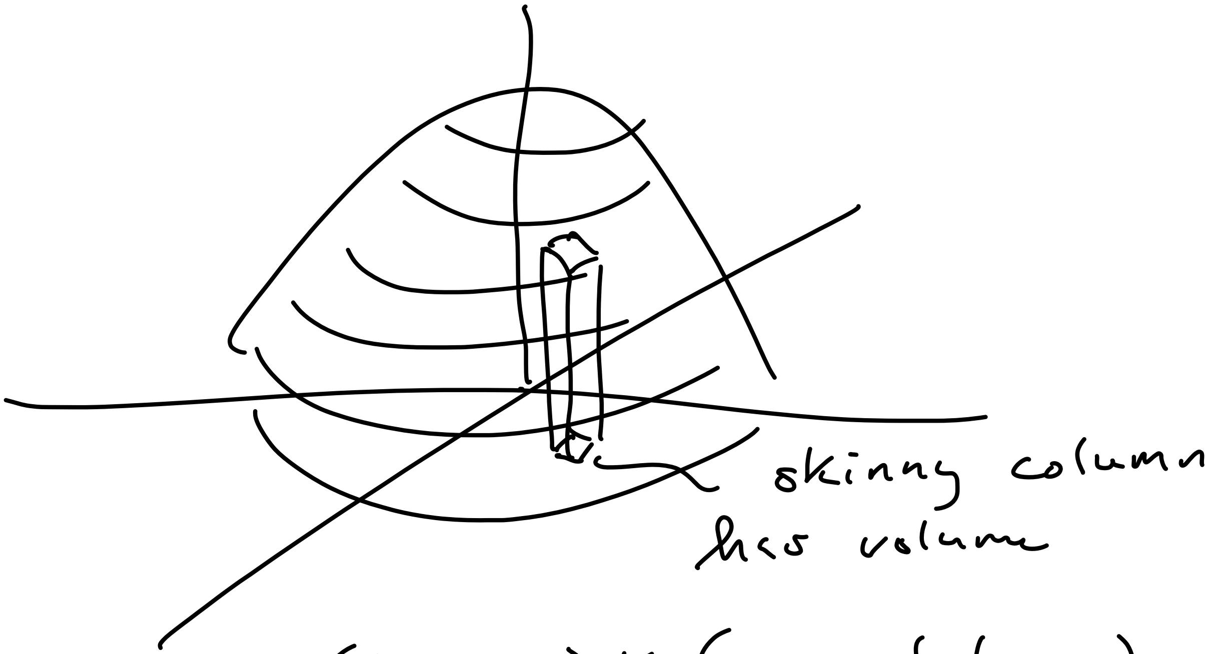
$$= \int_0^1 0 \, dy = 0.$$

So the "volume" of 3D region between rectangle

$$-1 \leq x \leq +1$$
$$0 \leq y \leq 1$$

and the surface $z = xy^2$ is zero. How can a volume be zero? Volume BELOW the x,y plane counts as negative.

Harder Example: Compute volume between x,y plane & parabolic dome $z = 1 - x^2 - y^2$:



(height) × (area of base)

$(1 - x^2 - y^2)\,dx\,dy$.

$$\text{Volume} = \iint (1 - x^2 - y^2)\,dx\,dy.$$

limits of integration ?

Need to sum over all points (x, y) in the unit disk.

$$? \le x \le ?$$
$$? \le y \le ?$$

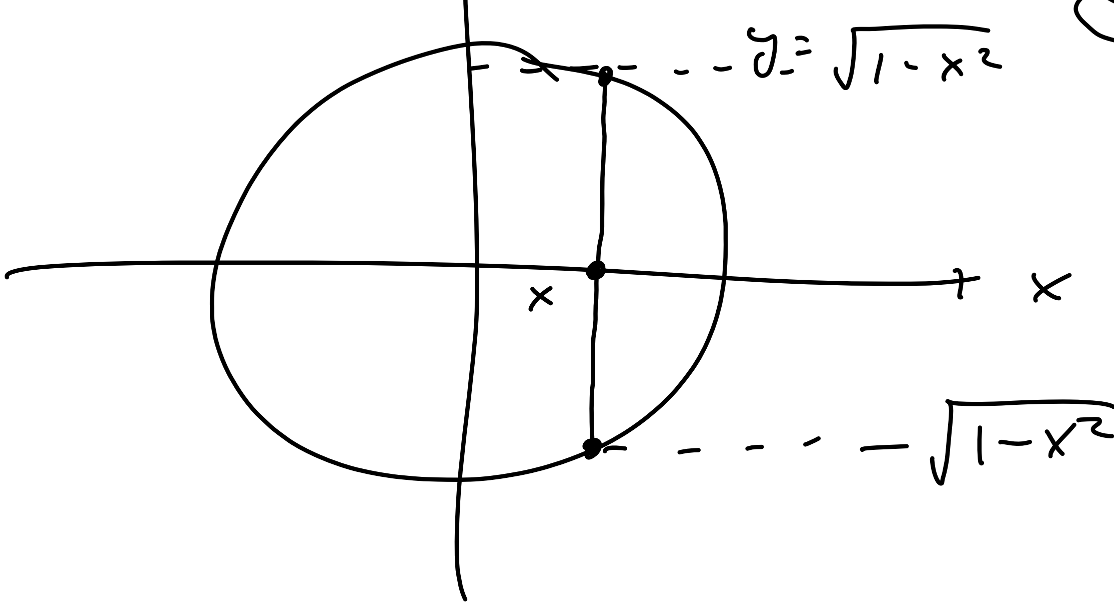
There are 2 ways to do it:

First let $-1 \le x \le +1$.

Then for each value of x, let

$$-\sqrt{1-x^2} \leq y \leq +\sqrt{1-x^2}$$

$y^2 = 1 - x^2$ $\quad x^2+y^2=1$



This choice of parametrization forces the order of integration:

$$\text{vol} = \int_{x=-1}^{1} \left(\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} (1-x^2-y^2)\, dy \right) dx$$

$$= \int_{x=-1}^{1} \left[y - x^2 y - \frac{1}{3} y^3 \right]_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} dx$$

$$= 2\int (1-x^2)\sqrt{1-x^2} - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 \, dx$$

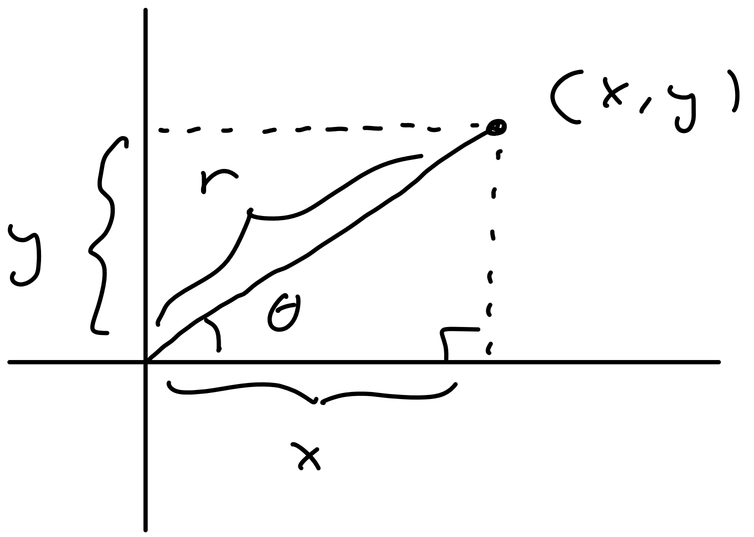
$$= 2\int_{x=-1}^{+1} \frac{2}{3}(1-x^2)^{3/2} \, dx$$

This is bad & my compter says answer is $\pi/2$.

Since the answer is nice, there must be an easier way to do this.

Polar / Cylindrical Coordinates.

When we integrate over a region of x,y plane with rotational symmetry it's better to use polar coordinates:

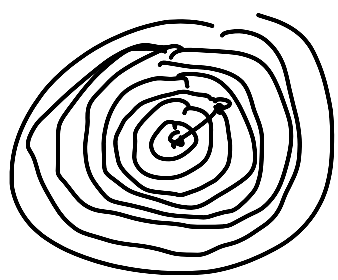


$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$
$$y = r\sin\theta$$

just constants

Parametrize the unit disk:

$$0 \leq r \leq 1$$
$$0 \leq \theta \leq 2\pi.$$



Nice property of $z = (1 - x^2 - y^2)$.

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2\left(\cancelto{1}{\cos^2\theta + \sin^2\theta}\right)$$
$$= r^2$$

So

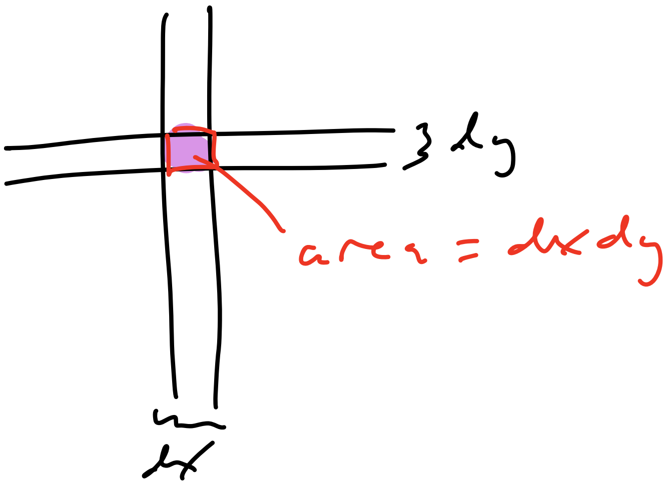
$$z = (1 - x^2 - y^2) = 1 - r^2.$$

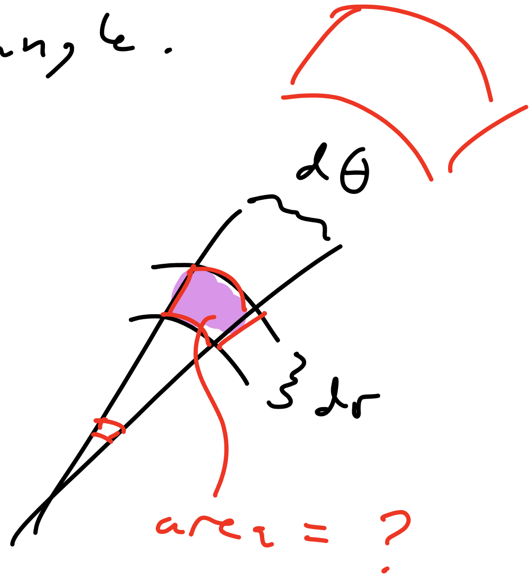
Idea:

$$\text{Vol} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (1-r^2)\,dr\,d\theta .$$

But this is not quite correct!

Reason: For tiny changes in r & θ we don't get a rectangle.

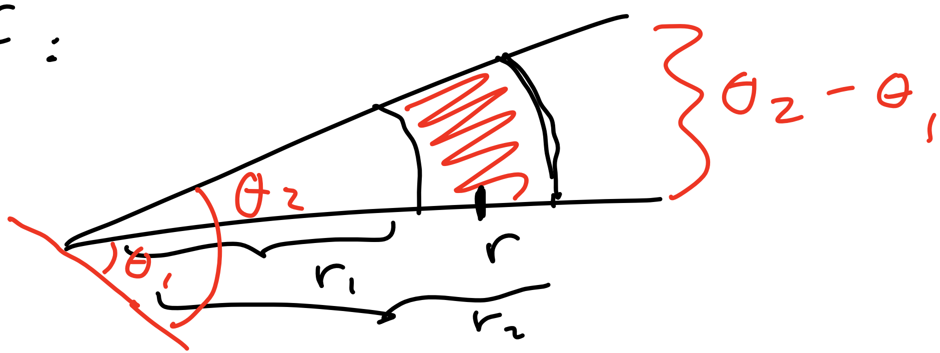




Theorem: Tiny region caused by tiny changes in r & θ has area

$$r\,dr\,d\theta .$$

Fake Proof:



Area:

$$\underbrace{(\pi r_2^2 - \pi r_1^2)}_{\text{area between circles}} \underbrace{(\theta_2 - \theta_1)/2\pi}_{\text{how much of the circle do you want.}}$$

Let $r_2 \to r_1$ & $\theta_2 \to \theta_1$

$r_2 - r_1 \to dr$ $\quad\quad \theta_2 - \theta_1 = d\theta$

$r_2, r_1 \longrightarrow r.$

Area:

$$\frac{1}{2}(r_2^2 - r_1^2)(\theta_2 - \theta_1)$$

$$= \frac{1}{2}\underbrace{(r_1 + r_2)}_{r+r}\underbrace{\cancel{(r_2 - r_1)}}_{dr}\underbrace{\cancel{(\theta_2 - \theta_1)}}_{d\theta}$$

$$= r\, dr\, d\theta.$$

As I said, it's a fake proof.

Just a heuristic.

Correct Computation:

vol of parabolic dome

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \underbrace{(1-r^2)}_{\text{height of skinny column}} \underbrace{r\,dr\,d\theta}_{\text{area of base of skinny column.}}$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{1} (r - r^3)\, dr \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} r^2 - \frac{1}{4} r^4 \right)\Bigg|_{r=0}^{r=1} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} - \frac{1}{4} \right) d\theta$$

$$= \int_{0}^{2\pi} \frac{1}{4}\, d\theta = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2} \checkmark$$