HW3 Problem 3:

Convert θ to radians.

HW3 Problem 2:

For any resonable function $f(x,y)$ we will have

$$f_{xy} = f_{yx}$$

$$\frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{df}{dy}\right)$$

There are some pathological counterexamples, but we can ignore them.

Hint: $f(x,y)$, $x(r,\theta)$, $y(r,\theta)$.

f is (indirectly) a function of r & θ. Like to compute derivatives

$$f_r, f_\theta, f_{rr}, f_{r\theta}, f_{\theta\theta}.$$

For these we need chain rule.

$$f_r = f_x \cdot x_r + f_y \cdot y_r$$

$$f_{rr} = \frac{d}{dr}\left(f_x \cdot x_r + f_y \cdot y_r \right)$$

$$= \underbrace{\frac{d}{dr}\left(f_x \cdot x_r\right)}_{\text{product rule}} + \underbrace{\frac{d}{dr}\left(f_y \cdot y_r\right)}_{\text{product rule}}$$

$$= f_{xr} \cdot x_r + f_x \cdot x_{rr}$$

$$+ f_{yr} \cdot y_r + f_y \cdot y_{rr}$$

Also need to simplify f_{xr}, f_{yr}.

Well, $f_x(x,y)$ is a function of 2 variables x & y, so

$$f_{xr} = f_{xx} \cdot x_r + f_{xy} \cdot y_r .$$

$$\frac{df_x}{dr} = \frac{df_x}{dx} \cdot \frac{dx}{dr} + \frac{df_x}{dy} \cdot \frac{dy}{dr} .$$

More on Linear Approximation.

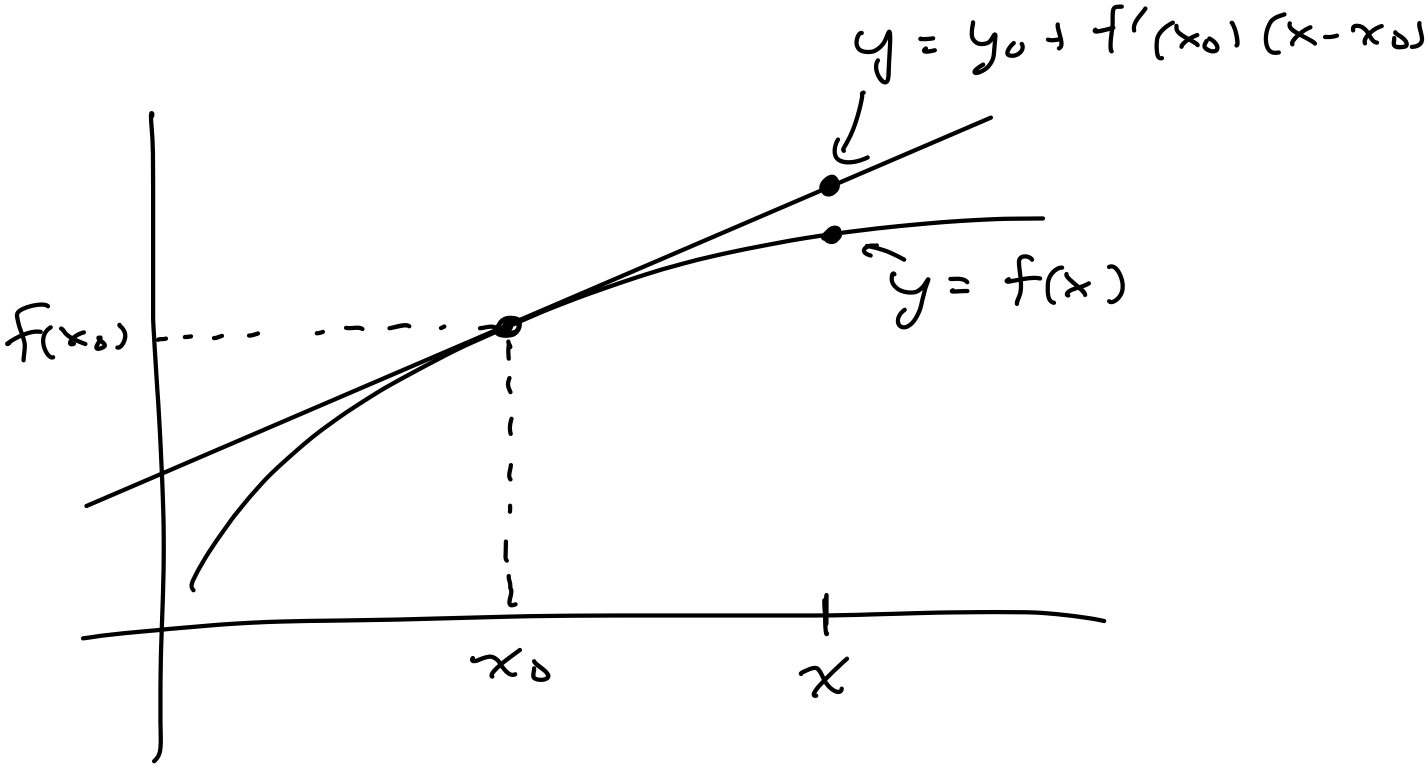
Given $f: \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ we have a Taylor expansion of $f(x)$ near $x = x_0$:

$$
\begin{aligned}
f(x) = f(x_0) &+ f'(x_0)(x - x_0) \\
&+ \frac{1}{2} f''(x_0)(x - x_0)^2 \\
&+ \frac{1}{6} f'''(x_0)(x - x_0)^3 \\
&\vdots \\
&+ \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \\
&\vdots
\end{aligned}
$$

Get an approximation by stopping the series early. Linear approx:

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0)}$$

Picture:



Equation of the tangent line.

slope $f'(x_0)$, point (x_0, y_0).

Point-Slope form:

$$y - y_0 = f'(x_0)(x - x_0)$$

$$y = y_0 + f'(x_0)(x - x_0)$$

Why do we care?

$$f(x) \approx \underset{f(x_0)}{\cancel{y_0}} + f'(x_0)(x - x_0)$$

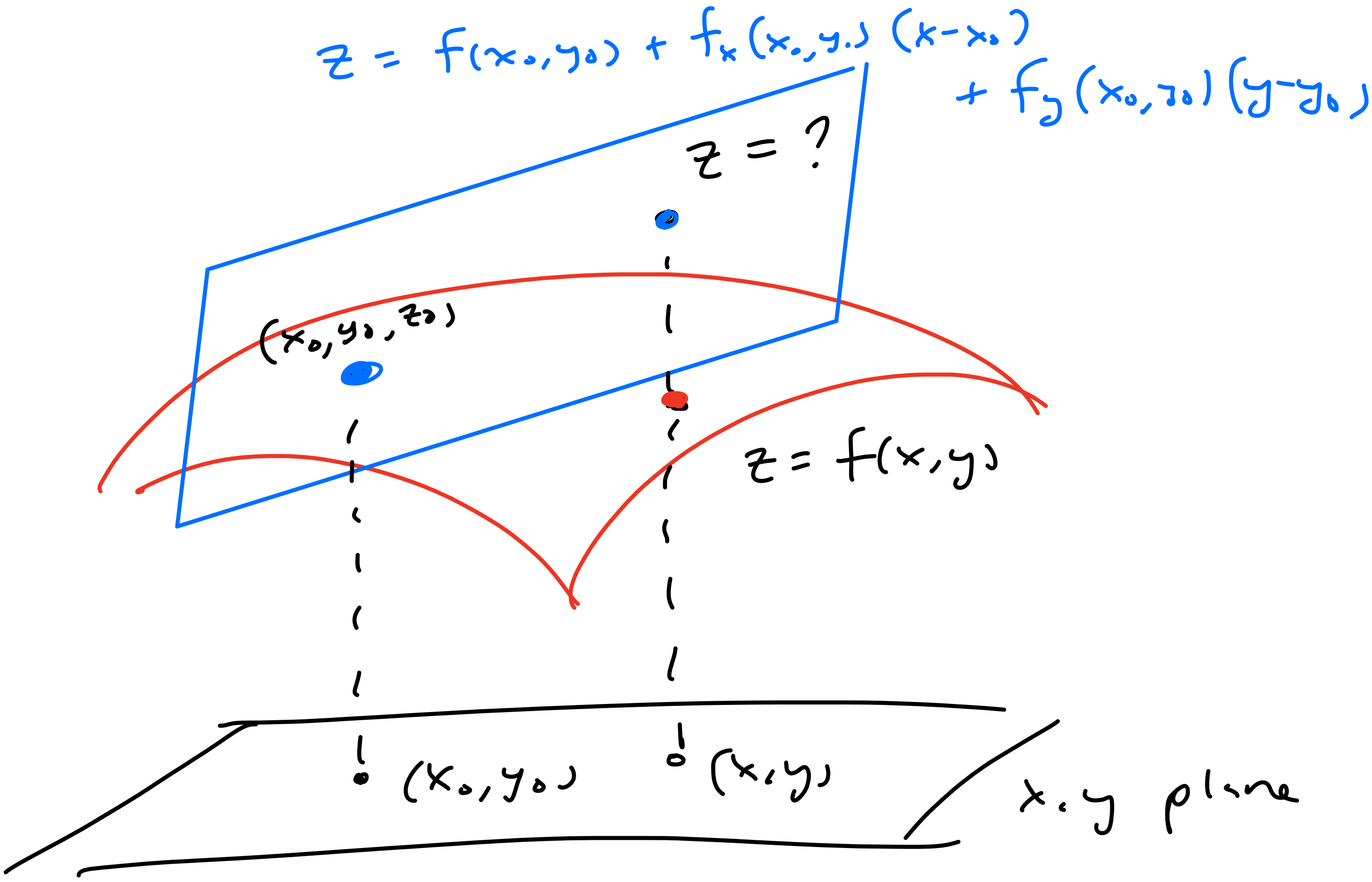
EASIER TO CALCULATE!

Function of 2 variables $f(x,y)$ has a Taylor expansion near $(x,y) = (x_0, y_0)$:

$$
\begin{aligned}
f(x,y) = {} & f(x_0,y_0) \\
& + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\
& + \frac{1}{2}\Big[f_{xx}(x_0,y_0)(x-x_0)^2 \\
& \qquad + 2 f_{xy}(x_0,y_0)(x-x_0)(y-y_0) \\
& \qquad + f_{yy}(x_0,y_0)(y-y_0)^2 \Big] \\
& + \frac{1}{6}\Big[\text{stuff like} \\
& \qquad f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 \cdots \Big] \\
& \vdots
\end{aligned}
$$

Cut it off to get linear approx:

$$
\begin{aligned}
\underset{z}{f(x,y)} \approx {} & \underset{z_0}{f(x_0,y_0)} + f_x(x_0,y_0)(x-x_0) \\
& + f_y(x_0,y_0)(y-y_0).
\end{aligned}
$$



$$z_0 = f(x_0,y_0)$$

Equation of Tangent Plane:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Linear Approx:

$$z \approx z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Language: $y = f(x)$

$$y - y_0 \approx f'(x_0)(x - x_0)$$

$$\Delta y \approx f'(x_0)\,\Delta x$$

$$\Delta y \approx \frac{dy}{dx}\,\Delta x$$

$$dy = \frac{dy}{dx}\cdot dx$$ Chain Rule

$$\left(\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}\right)$$

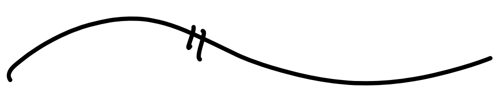
$z = f(x,y)$

$$z - z_0 \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

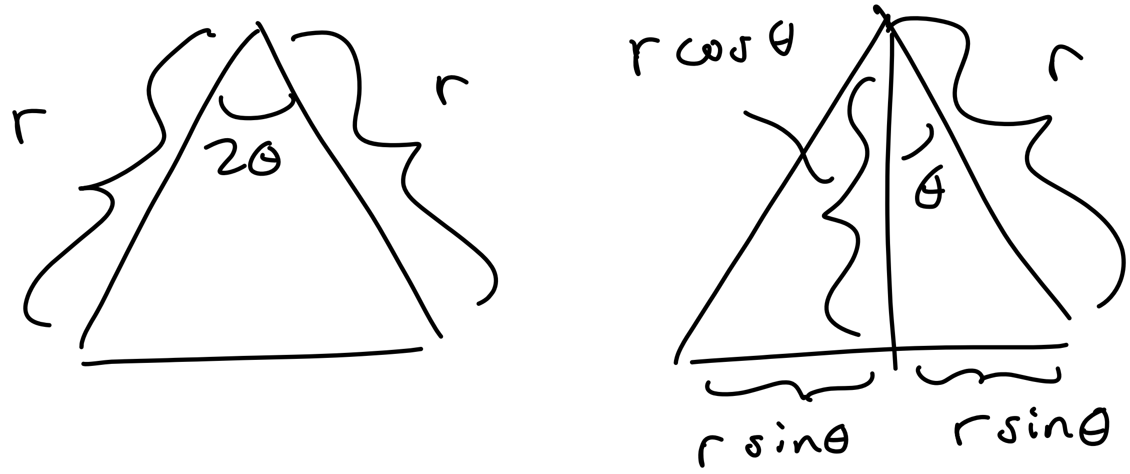
$$\Delta z \approx \frac{df}{dx}\cdot\Delta x + \frac{df}{dy}\cdot\Delta y.$$

$$dz = \frac{df}{dx}\,dx + \frac{df}{dy}\,dy.$$

Chain Rule!



Example: Estimate area of isoceles triangle



$$A(r,\theta) = \frac{1}{2}\,\text{base}\cdot\text{height}$$

$$= \frac{1}{2}(2r\sin\theta)(r\cos\theta)$$

$$= r^2 \sin\theta\cos\theta$$

$$= \frac{r^2}{2}\sin(2\theta)$$

$$A_r = r\sin(2\theta)$$

$$A_\theta = \frac{r^2}{2}\cos(2\theta)\cdot 2$$

$$= r^2\cos(2\theta).$$

Linear Approximation

$$dA = A_r\, dr + A_\theta\, d\theta$$

$$= r \sin(2\theta)\, dr + r^2 \cos(2\theta)\, d\theta .$$

OR. Near r_0, θ_0,

$$A(r,\theta) - A(r_0,\theta_0) \approx$$

$$+ r_0 \sin(2\theta_0)(r - r_0)$$

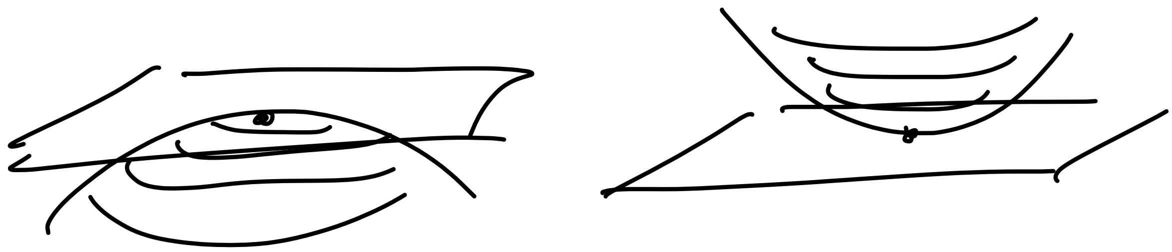
$$+ r_0^2 \cos(2\theta_0)(\theta - \theta_0)$$

Unconstrained Optimization.

Find local max/min of a scalar field $f(x,y)$.

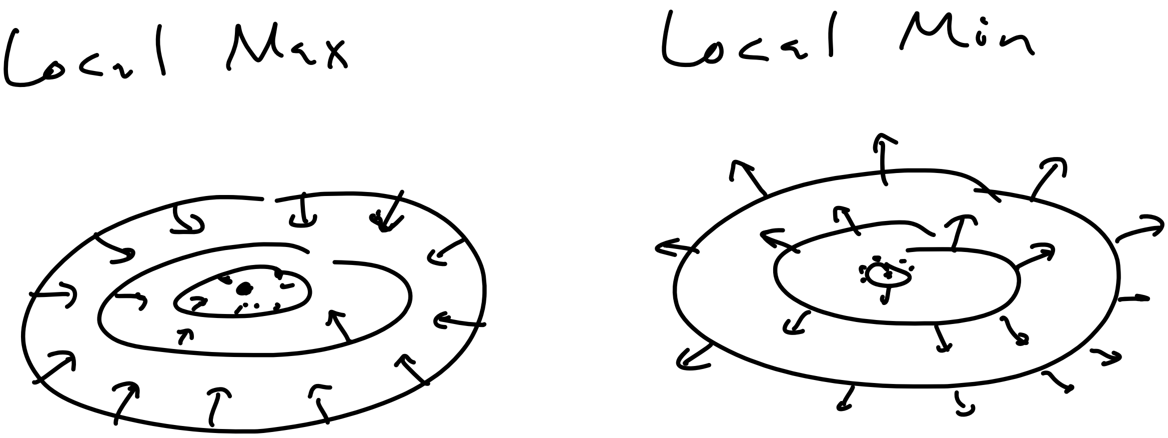
If we look at the graph $z = f(x,y)$ these are places where the tangent plane is horizontal:

Local Max



This happens when $\nabla f(x,y)$ vanishes, i.e., equals $\langle 0,0 \rangle$.

In 2D:



Example: $f(x,y) = x^2 + y^2$.

$\nabla f(x,y) = \langle 2x, 2y \rangle = \langle 0, 0 \rangle$.

"Critical point" $(x,y) = (0,0)$.

Turns out to be a min.

(Parabolic shaped Valley).

Example: $f(x,y) = xy$.

$$\nabla f(x,y) = \langle y, x \rangle = \langle 0, 0 \rangle$$

critical point $(x,y) = (0,0)$

Min or Max? NO!

It's a "saddle point".

[see Geogebra]

Example: $f(x,y) = x^2 y$

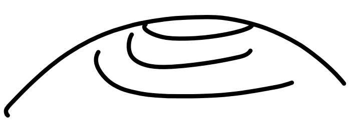
$$\nabla f(x,y) = \langle 2xy, x^2 \rangle = \langle 0, 0 \rangle.$$

<u>Every point $\langle 0, y \rangle$ is critical!</u>

These turn out to be degenerate critical points.

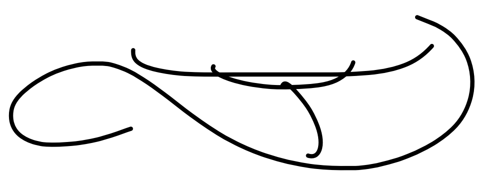
[See Geogebra].

Summary:

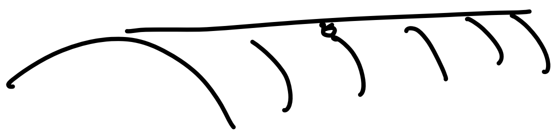


max

min



saddle.



degenerate
(flat in some
direction)

The "second derivative test"
lets us distinguish between these.

Hessian Matrix of $f(x,y)$:

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Hessian determinant

$$\det(Hf) = f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx}$$
$$= f_{xx} \cdot f_{yy} - f_{xy}^2.$$

Theorem:

Let (x_0, y_0) be critical:

$$\nabla f(x_0, y_0) = \langle 0, 0 \rangle$$

i.e. $f_x(x_0, y_0) = 0$ & $f_y(x_0, y_0) = 0$

Then:

- If $\det(Hf)(x_0,y_0) < 0$ then (x_0,y_0) is a <u>saddle</u> point.

- If $\det(Hf)(x_0,y_0) = 0$ called "degenerate". [Too flat for this test to work ...]

- If $\det(Hf)(x_0,y_0) > 0$ then (x_0,y_0) is local max or min
  - Max when $f_{xx}(x_0,y_0) < 0$
  - Min when $f_{xx}(x_0,y_0) > 0$

Example: $f(x,y) = x^2 - x^4 - y^2 + 1$

$$\nabla f(x,y) = \langle 2x - 4x^3, -2y \rangle = \langle 0,0 \rangle$$

$$\begin{cases} 2x - 4x^3 = 0 \\ -2y = 0 \longrightarrow y = 0 \end{cases}$$

ALWAYS

$$2x(1-2x^2) = 0$$

$$x = 0 \quad \text{or} \quad x = \pm 1/\sqrt{2}.$$

3 critical points:

$$(0,0), \quad \left(\frac{1}{\sqrt{2}}, 0\right), \quad \left(-\frac{1}{\sqrt{2}}, 0\right)$$

$$f_{xx} = (2x - 4x^3)_x = 2 - 12x^2$$

$$\left.\begin{aligned} f_{xy} &= (2x - 4x^3)_y = 0 \\ f_{yx} &= (-2y)_x = 0 \end{aligned}\right\} \text{SAME} \ :)$$

$$f_{yy} = (-2y)_y = -2$$

Hessian Matrix

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2-12x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\det(Hf) = (2-12x^2)(-2) - 0$$

$$= -2 + 12x^2.$$

Evaluate $\det(Hf)$ at our critical points:

• $\det(Hf)(0,0) = -2 + 12 \cdot 0^2$
$= -2 < 0$.
SADDLE.

• $\det(Hf)\left(\frac{1}{\sqrt{2}}, 0\right) = -2 + 12\left(\frac{1}{\sqrt{2}}\right)^2$
$= -2 + 6 = 4 > 0$

• $\det(HF)\left(-\frac{1}{\sqrt{2}}, 0\right) = -2 + 12\left(-\frac{1}{\sqrt{2}}\right)^2$
$= -2 + 6 = 4 > 0$

Local max or min ...

To see which we look at f_{xx}.

$F_{xx}(x,y) = 2 - 12x^2$.

$F_{xx}\left(\frac{1}{\sqrt{2}}, 0\right) = 2 - 12\left(\frac{1}{\sqrt{2}}\right)^2$
$= 2 - 6 = -4 < 0$
MAX!

$f_{xx}\left(-\frac{1}{\sqrt{2}}, 0\right) = 2 - 12\left(-\frac{1}{\sqrt{2}}\right)^2$
$= 2 - 6 = -4 < 0$
MAX!

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