Final Project due Fri: 11:59 PM
on Blackboard.

Bonus Lecture:

Stokes Theorem $\longrightarrow$ Green's Theorem

3D $\qquad\qquad\qquad$ 2D.

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$.

$\text{curl}(\vec{F}) = Q_x(x,y) - P_y(x,y)$.

[detects c.c.w. rotation]

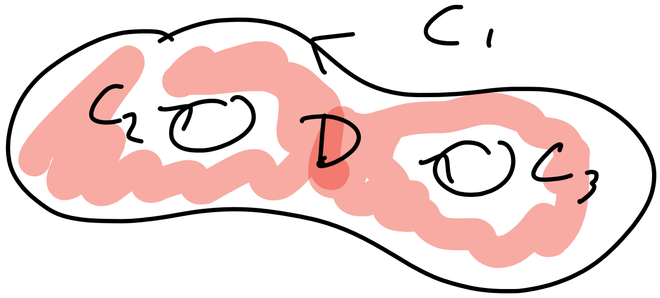
Green (special case of Stokes):

$$\iint_D \text{curl}(\vec{F}) = \int_{\partial D} \vec{F}$$

$$\iint (Q_x - P_y)\,dx\,dy = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

for parametrization $\vec{r}(t)$ of bdry curve.

The boundary curve can have multiple components:



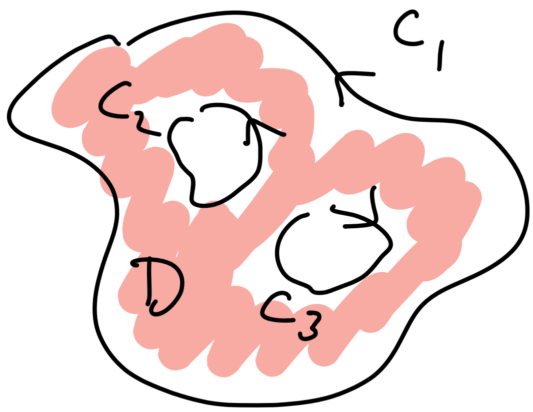

$$\partial D = C_1 + C_2 + C_3$$

[Rule: $D$ is "to the left" of $\partial D$.]

Reason for notation:

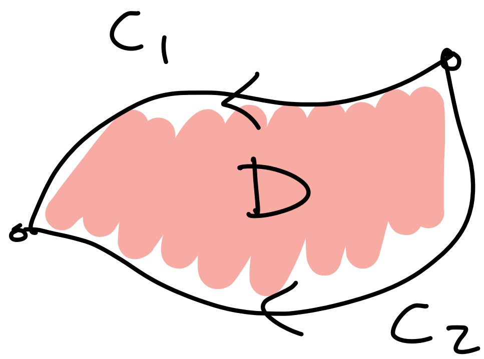
$$\int_{C_1+C_2+C_3} \vec{F} = \int_{C_1} \vec{F} + \int_{C_2} \vec{F} + \int_{C_3} \vec{F}$$

We can also reverse orientation:




$$\partial D = C_1 - C_2 + C_3$$

[$C_2$ is backwards: it has $D$ "on the right"]




$$\partial D = C_1 - C_2$$

e.g. IF $\operatorname{curl}(\vec{F}) = 0$ on $D$. Then

$$0 = \iint_D \operatorname{curl}(\vec{F}) = \int_{C_1 - C_2} \vec{F}$$

$$= \int_{C_1} \vec{F} - \int_{C_2} \vec{F}$$

$$\Longrightarrow \int_{C_1} \vec{F} = \int_{C_2} \vec{F}.$$

Summary:

$\operatorname{curl}(\vec{F}) = 0$

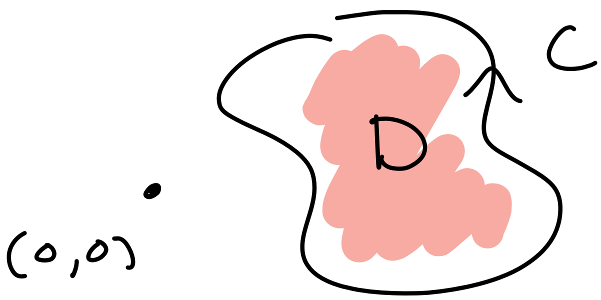
$\Longrightarrow \int_C \vec{F}$ only depends on endpoints of $C$, not the shape.

Example: $\vec{F}(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$.

$P = -y/x^2+y^2 \rightsquigarrow P_y = y^2-x^2/(x^2+y^2)^2$

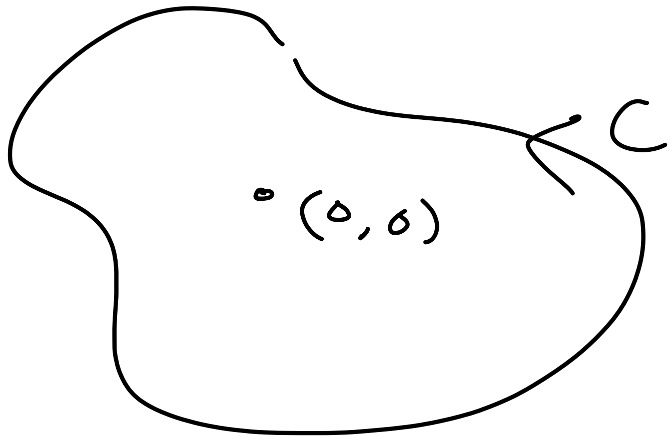
$Q = x/x^2+y^2 \rightsquigarrow Q_x = y^2-x^2/(x^2+y^2)^2$.

So $\operatorname{curl}(\vec{F}) = Q_x - P_y = 0$, when it is defined. It's not defined at $(0,0)$. If a loop $C$ does not contain $(0,0)$




then $\int_C \vec{F} = \iint_D \operatorname{curl}(\vec{F}) = \iint_D 0 = 0$.

What about a loop that contains $(0,0)$?



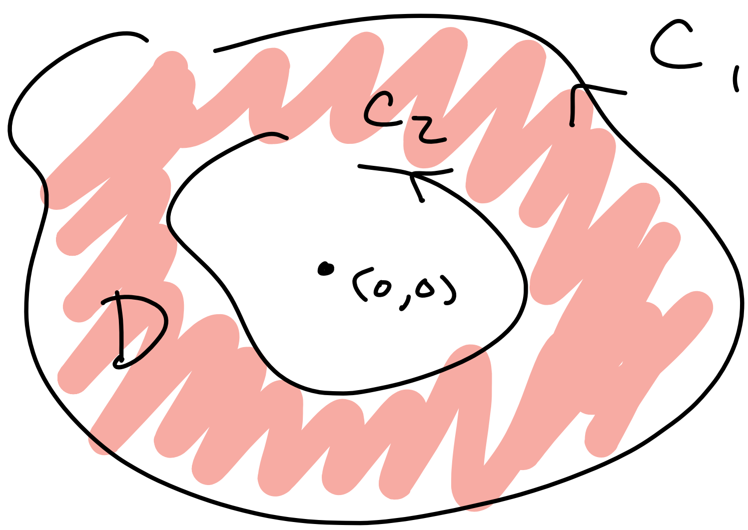

$$\int_C \vec{F} = ?$$

Claim: $\int_C \vec{F} = 2\pi$,

independent of the shape of $C$.

Proof has 2 steps:

(1) Any two such loops have the same integral:




$$\partial D = C_1 - C_2$$

$$0 = \iint_D \operatorname{curl}(\vec{F}) = \int_{C_1 - C_2} \vec{F}$$

$$= \int_{C_1} \vec{F} - \int_{C_2} \vec{F}$$

$$\Longrightarrow \int_{C_1} \vec{F} = \int_{C_2} \vec{F}.$$

(2) So pick the easiest curve.

$$\vec{r}(t) = \langle \cos t, \sin t \rangle.$$

$$\int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \left\langle \frac{-\sin t}{\underbrace{\cancel{\cos^2 t + \sin^2 t}}_{1}}, \frac{\cos t}{\underbrace{\cancel{\cos^2 t + \sin^2 t}}_{1}} \right\rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \quad \checkmark$$

Summary: $\vec{F}(x,y) = \frac{1}{x^2+y^2} \langle -y, x \rangle.$

$$\oint_C \vec{F} = \begin{cases} 0 & C \text{ does not contain } (0,0) \\ 2\pi & C \text{ goes around } (0,0) \text{ once in c.c.w. direction} \\ 2\pi k & C \text{ goes around } (0,0)\ k \text{ times in c.c.w. direction.} \end{cases}$$

Flux form of Green's Theorem.

Let $\vec{F} = \langle P, Q \rangle$

$\vec{G} = \langle u, v \rangle = \langle -Q, P \rangle.$

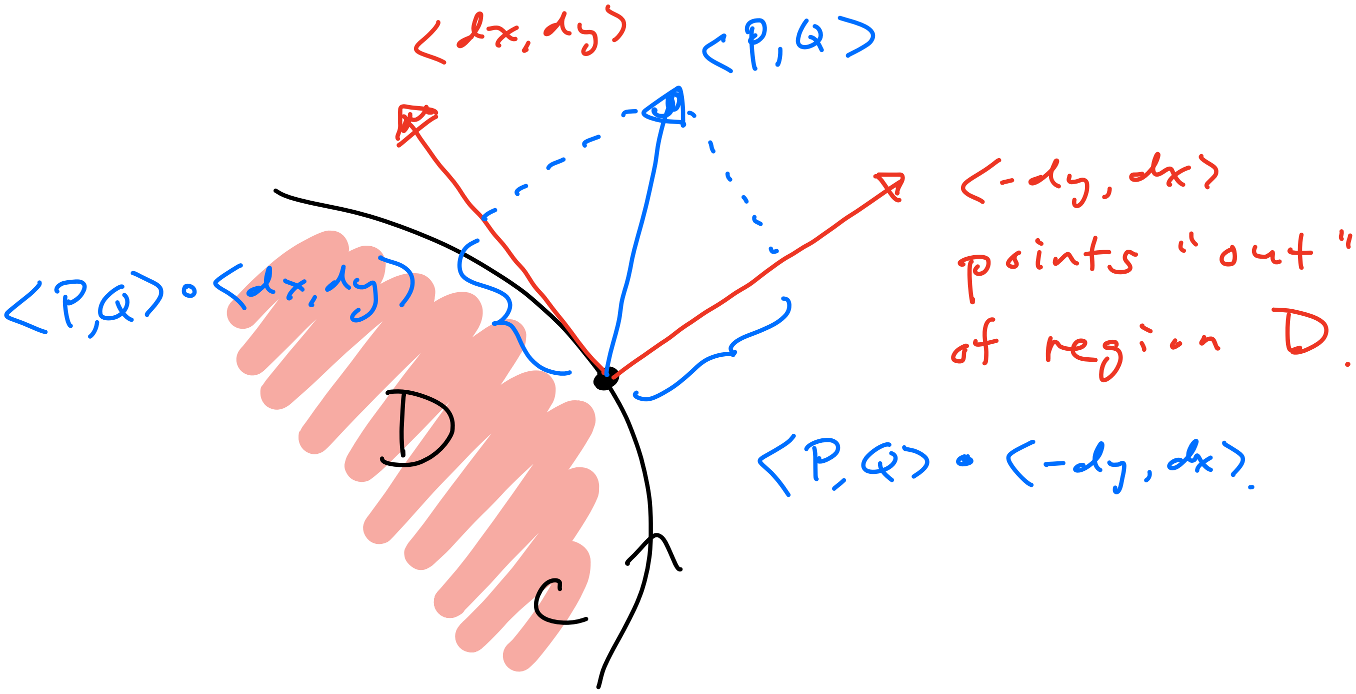
Apply Green to $\vec{G}$.

$$\iint_D (V_x - U_y)\, dx\, dy = \int_{\partial D} \langle u, v \rangle \cdot \langle dx, dy \rangle.$$

$$\iint (P_x + Q_y)\, dx\, dy = \int \langle -Q, P \rangle \cdot \langle dx, dy \rangle$$

$$= \int \langle P, Q \rangle \cdot \langle -dy, dx \rangle.$$

What?




So $\displaystyle\int_C \langle P, Q\rangle \cdot \langle -dy, dx\rangle$

measures how much the vector field $\langle P, Q\rangle$ points "out of" the region D. Called "Flux".

Cleaner Notation:

$$\underbrace{\iint_D (Q_x - P_y)\,dx\,dy}_{\text{amount of curling in } D} = \underbrace{\int_{\partial D} \vec{F} \cdot \vec{T}}_{\text{amount } \vec{F} \text{ points along } \partial D.}$$

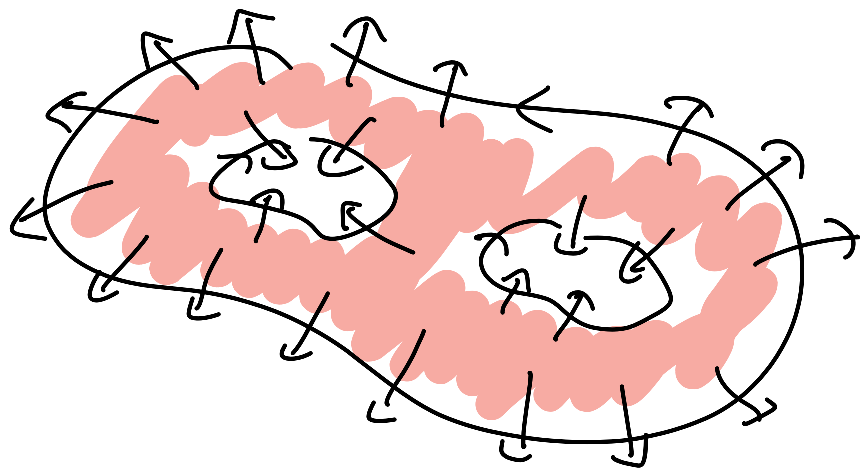
little tangent vector

$$\iint_D (P_x + Q_y)\,dx\,dy = \int_{\partial D} \vec{F} \cdot \vec{N}$$

little normal vector.

amount that $\vec{F}$ expands inside D

amount that $\vec{F}$ flows across $\partial D$.



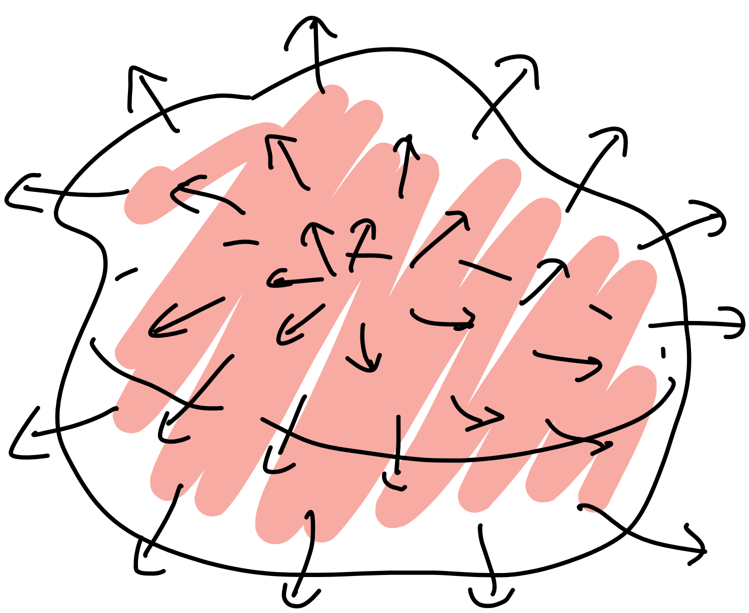
Moving from 2D to 3D: Green's Theorem becomes two different theorems.

$$\begin{tikzcd} & \text{Green} \arrow[dl, "\text{circulation}"'] \arrow[dr, "\text{flux}"] & \\ \text{Stokes} & & \text{Gauss (Divergence Theorem)} \end{tikzcd}$$

We've seen Stokes.

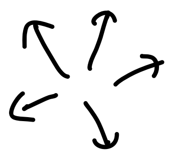
Now: Gauss' Theorem

$$\underbrace{\iiint_V \nabla\cdot\vec{F}}_{\text{expansion of } \vec{F} \text{ in a volume } V} = \underbrace{\iint_{\partial V} \vec{F}\cdot\vec{N}}_{\text{flow of } \vec{F} \text{ across boundary surface } \partial V.}$$

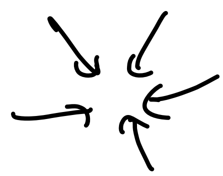


$$\nabla \circ \langle P, Q, R \rangle = P_x + Q_y + R_y$$

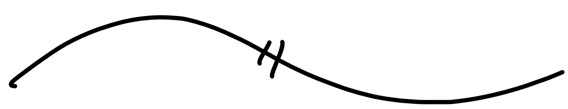
Gauss tells us what this strange formula has to do with "expansion/contraction".



$\nabla \circ \vec{F} > 0$



$\nabla \circ \vec{F} < 0$



Application to Gravity.

Let $\vec{F}(x,y,z)$ be the gravitational force acting on a particle of mass $m$ at point $(x,y,z)$, due to some mass distribution $\rho(x,y,z)$.

Gauss' Law:

assume gravitational constant = 1

$$\nabla \cdot \vec{F} = -4\pi m \, \rho(x,y,z)$$

This is equivalent to (and more useful than) Newton's universal gravitation. It is particularly useful when dealing with a spherically symmetric density $\rho$.

Suppose

$$M = \text{total mass} = \iiint \rho \, dV$$

Suppose

$$\vec{F}(\vec{r}) = \underbrace{F(r)}_{\text{some scalar function of } r = \|\vec{r}\|} \frac{\vec{r}}{\|\vec{r}\|}$$

i.e. the force is the same in all directions, only depends on the distance from $(0,0,0)$.

Then Divergence Theorem says

$$\underset{\substack{\text{ball} \\ \text{radius } r}}{\iiint} \nabla \cdot \vec{F} = \underset{\substack{\text{sphere} \\ \text{radius } r}}{\iint} \underbrace{\vec{F} \cdot \vec{N}}$$

we can use $\vec{N} = \frac{\vec{r}}{\|\vec{r}\|}$

this is the component of $\vec{F}$ normal to surface of the sphere.

$$\iiint -4\pi m \rho = \iint F(r)$$

$$-4\pi m M_r = F(r) \, 4\pi r^2$$

how much mass inside ball of radius $r$.

surface area of sphere

Three interesting Examples:

- Point particle at $(0,0,0)$ mass $M$.

$F(r) = -Mm/r^2$ (Newton).

- Solid sphere radius $R$.

$$F(r) = \begin{cases} -Mm/r^2 & r \geq R \\ -\dfrac{Mm}{R^3} r & r < R \end{cases}$$

- Empty shell radius $R$ with all mass $M$ on its boundary

$$F(r) = \begin{cases} -Mm/r^2 & r \geq R \\ 0 & r < R. \end{cases}$$

Inside a massive spherical shell you fell <u>no</u> gravity.

Called "Newton's shell theorem", proved by him using complicated argument.

Gauss' Law & Divergence Thm make it "almost obvious" ;)