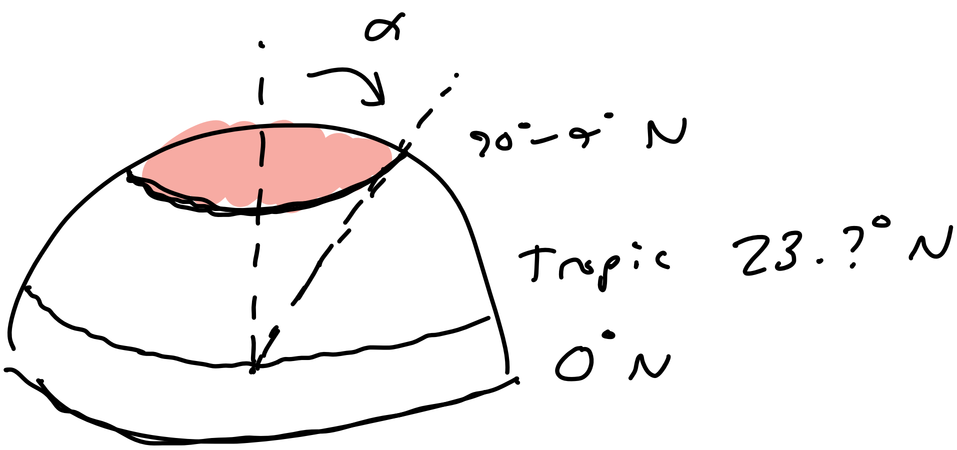
Today: HW 5 Discussion.

Tomorrow: Quiz 5 & Bonus Lecture.

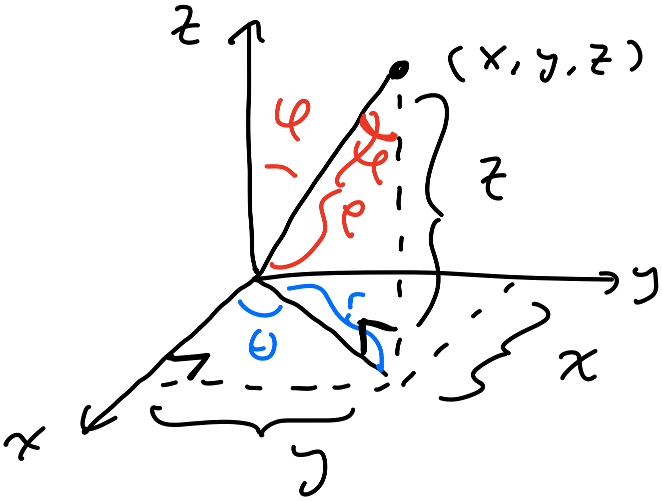
Fri: Final Project Due.

Problem 1: Cap of a sphere:



Parametrize this 2D region:

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle.$$



$x = r\cos\theta$

$y = r\sin\theta$

$z = \rho\cos\varphi$

$r = \rho\sin\varphi$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

Unit ~~Ball~~ Sphere :

$\cancel{0 \le \rho \le 1}$ $\quad \rho = 1$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \cancel{\pi}\ \alpha$ $\quad$ CAP

$$\vec{F}(\theta, \varphi) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$$

Area Stretch Factor $\|\vec{r}_\theta \times \vec{r}_\varphi\|$:

$$\vec{r}_\theta = \langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0 \rangle$$

$$\vec{r}_\varphi = \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi \rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi = \langle -\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta,$$
$$-\sin\varphi\cos\varphi \sin^2\theta$$
$$-\sin\varphi\cos\varphi \cos^2\theta \rangle$$

$$= \langle -\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi \rangle$$

$$\|\vec{r}_\theta \times \vec{r}_\varphi\|^2 =$$

$$\sin^4\varphi \cos^2\theta + \sin^4\varphi \sin^2\theta + \sin^2\varphi \cos^2\varphi$$

$$= \sin^4\varphi + \sin^2\varphi \cos^2\varphi$$

$$= \sin^2\varphi \, (\cancel{\sin^2\varphi + \cos^2\varphi})$$

$$= \sin^2\varphi$$

$$\|\vec{r}_\theta \times \vec{r}_\varphi\| = |\sin\varphi|$$

$$= \sin\varphi \qquad (0 \le \varphi \le \pi)$$

The area is

$$\iint_{\text{cap}} 1 \, dA = \iint_{\text{cap}} \overbrace{\|\vec{r}_\theta \times \vec{r}_\varphi\| \, d\theta \, d\varphi}^{dA}$$

$$= \iint \sin\varphi \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\alpha} \sin\varphi \, d\varphi$$

$$= 2\pi \left[-\cos\varphi \right]_0^{\alpha}$$

$$= 2\pi \left[-\cos\alpha + \cos 0 \right]$$

$$= 2\pi \left[1 - \cos\alpha \right].$$

Special Cases:

$\alpha = 0 \longrightarrow \text{area} = 0$ ✓

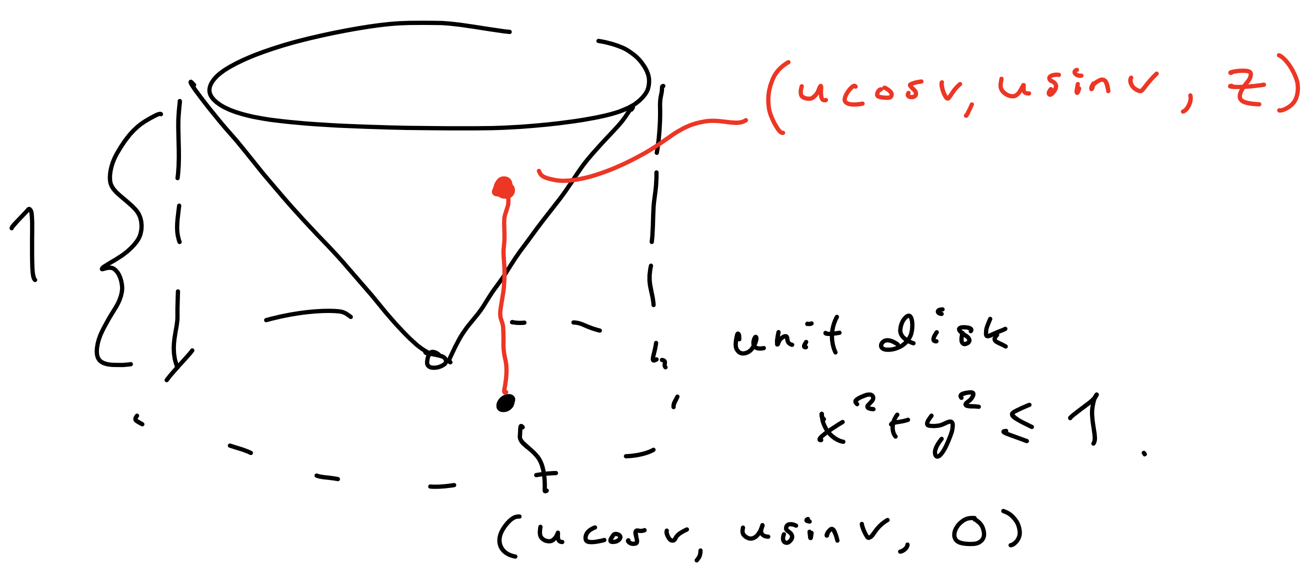
$\alpha = \pi/2 \longrightarrow \text{area} = 2\pi [1 - 0]$

$= 2\pi$ ✓

$\alpha = \pi \longrightarrow \text{area} = 2\pi [1 - (-1)]$

$= 4\pi$ ✓

Problem 2: $z^2 = x^2 + y^2$



$$z = \sqrt{x^2+y^2} = \sqrt{u^2} = u.$$

Parametrization:

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u \rangle.$$

$$0 \le u \le 1$$
$$0 \le v \le 2\pi.$$

$$\vec{r}_u = \langle \cos v,\ \sin v,\ 1 \rangle$$
$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -u\cos v,\ -u\sin v,\ \overset{u}{\cancel{u\cos^2 v + u\sin^2 v}} \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\|^2 = \underset{u^2}{\cancel{u^2\cos^2 v + u^2\sin^2 v}} + u^2$$
$$= 2u^2$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{2}\,u \qquad (u \ge 0)$$

Area:

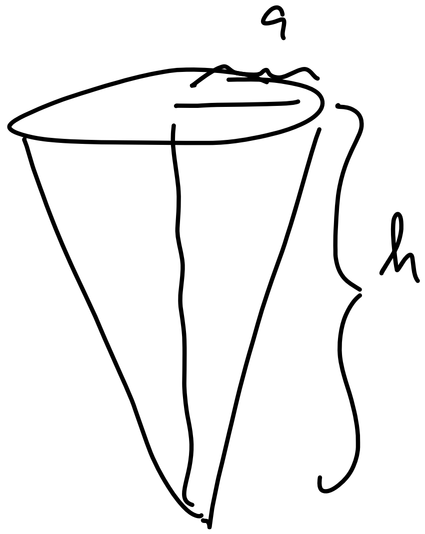
$$\iint_{\text{cone}} 1\,dA = \iint 1\,\overset{dA}{\|\vec{r}_u \times \vec{r}_v\|\,du\,dv}$$
$$= \iint \sqrt{2}\,u\,du\,dv$$

$$= \sqrt{2} \int_0^{2\pi} dv \int_0^1 u\,du$$

$$= \sqrt{2} \cdot 2\pi \cdot (1/2)$$

$$= \sqrt{2} \cdot \pi .$$

Remark:



surface area

$$= \pi a \sqrt{h^2 + a^2}$$

In our case

$a = 1$
$h = 1$

Problem 3: KE + PE.

$$\vec{F}(x,y,z) = \underbrace{\langle 0, 0, -mg \rangle}_{\text{constant}}.$$

$$\vec{F} = m\vec{r}''(t)$$

$$\langle 0, 0, -mg\rangle = m\langle x''(t), y''(t), z''(t)\rangle.$$

$$\vec{r}''(t) = \underbrace{\langle 0, 0, -g\rangle}_{\text{constant.}}$$

$$\vec{r}'(t) = \langle c_1, c_2, -gt + c_3\rangle$$

$$\vec{r}'(0) = \langle c_1, c_2, c_3\rangle = \underbrace{\langle u, v, w\rangle}_{\text{Given.}}$$

$$\vec{r}'(t) = \langle u, v, -gt + w\rangle.$$

$$\vec{r}(t) = \langle ut + c_4, vt + c_5, -\tfrac{1}{2}gt^2 + wt + c_6\rangle$$

$$\vec{r}(0) = \langle c_4, c_5, c_6\rangle = \underbrace{\langle 0, 0, 0\rangle}_{\text{Given}}$$

Position at time t:

$$\vec{r}(t) = \langle ut, vt, -\tfrac{1}{2}gt^2 + wt\rangle.$$

Kinetic energy at time t:

$$KE(t) = \frac{1}{2}m\|\vec{r}'(t)\|^2$$

$$= \frac{1}{2}m\left(u^2 + v^2 + (-gt + w)^2\right)$$

$$= \frac{1}{2} m \left(u^2 + v^2 + g^2 t^2 - 2gtw + w^2 \right)$$

$$= \underbrace{\frac{1}{2} m \left(u^2 + v^2 + w^2 \right)}_{\substack{KE(0) \\ \frac{1}{2} m \|\vec{r}\,'(0)\|^2}} + \frac{1}{2} m g^2 t^2 - mgtw$$

NOW: Observe that $\vec{F}$ is a conservative vector field.

[Recall:

$$\vec{F} = -\nabla f \iff \nabla \times \vec{F} = \langle 0, 0, 0 \rangle.$$

Definitely true for constant vector fields.

Given $\vec{F} = \langle a, b, c \rangle$ constant.

Find $f(x, y, z)$ such that $\nabla f = \vec{F}$.

Answer:

$$f(x, y, z) = ax + by + cz \quad + \text{any constant}.$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle a, b, c \rangle. \quad]$$

In our case $\vec{F} = \langle 0, 0, -mg \rangle$

so $-f = 0x + 0y - mgz$. + const

$f = +mgz$ + const   take const 0 so $f(0,0,0) = 0$.

so $\vec{F} = -\nabla f$

Potential Energy at time t:

$$PE(t) = f(\vec{r}(t))$$

$$= f\left(ut, vt, -\tfrac{1}{2}gt^2 + wt\right)$$

$$= +mg\left(-\tfrac{1}{2}gt^2 + wt\right).$$

$$= -\tfrac{1}{2}mg^2t^2 + mgwt.$$

Why do we care? Because

$$KE(t) + PE(t)$$

$$= \underbrace{\tfrac{1}{2}m(u^2+v^2+w^2)}_{\text{independent of time.}}$$

Problem 4:

$$\vec{F}(x,y,z) = \langle y+z, x+z, x+y \rangle$$
$$\vec{G}(x,y,z) = \langle -y+z, x+z, x+y \rangle$$

(a): Curl of $\vec{F}$:

$$\nabla \times \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \times \langle P, Q, R \rangle.$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$P_y = P_z = 1$

$Q_x = Q_z = 1$

$R_x = R_y = 1$

$$= \langle 1-1, 1-1, 1-1 \rangle$$
$$= \langle 0, 0, 0 \rangle.$$

$$\nabla \times \vec{G} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

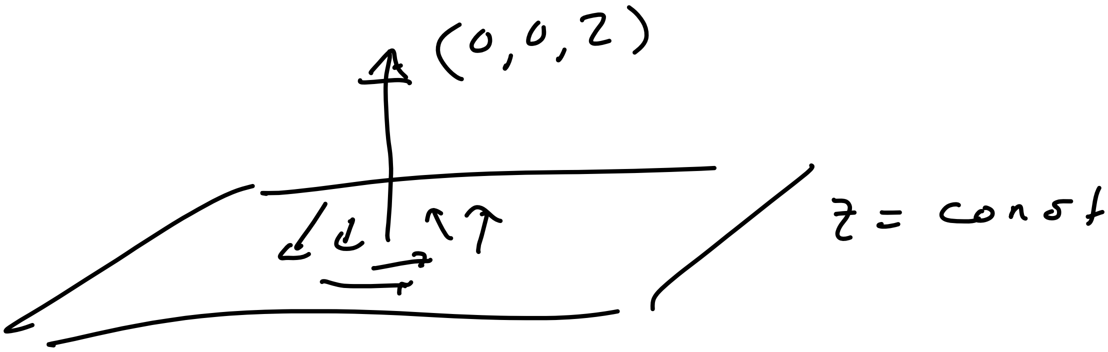
$P_y = -1$, $P_z = 1$

$Q_x = 1$, $Q_z = 1$

$R_x = 1$, $R_y = 1$

$$= \langle 1-1, 1-1, 1-(-1) \rangle$$
$$= \langle 0, 0, 2 \rangle.$$

Picture: $\vec{G}$ curls counterclockwise in the x,y-plane.



(b): Integrate $\vec{F}$ & $\vec{G}$ around the circle $\vec{r}(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{0} \rangle$.

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle.$$

$$\vec{F}(\vec{r}(t)) = \langle y+z, x+z, x+y \rangle$$

$$= \langle \sin t + 0, \cos t + 0, \sin t + \cos t \rangle.$$

$$\int_{\text{circle}} \vec{F} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t + \cos^2 t + 0 \; dt$$

$$= \int_0^{2\pi} \cos(2t)\, dt$$

$$= \left[\frac{1}{2} \sin(2t) \right]_0^{2\pi}$$

$$= \frac{1}{2} \sin(4\pi) - \frac{1}{2} \sin(0).$$

$$= 0 \quad \text{AS EXPECTED} \checkmark$$

$$\vec{G}(\vec{r}(t))$$

$$= \langle -\sin t + 0, \cos t + 0, \sin t + \cos t \rangle.$$

$$\int_{\text{circle}} \vec{G} = \int \vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int + \sin^2 t + \cos^2 t + 0 \; dt$$

$$= \int_0^{2\pi} 1 \, dt$$

$$= 2\pi \neq 0.$$

This is another proof that

$\vec{G}$ is not conservative.

[$\vec{G}$ will cause a particle to speed up when traveling around the circle.]

Since $\vec{F}$ is conservative we know there exists a scalar $f(x,y,z)$ such that

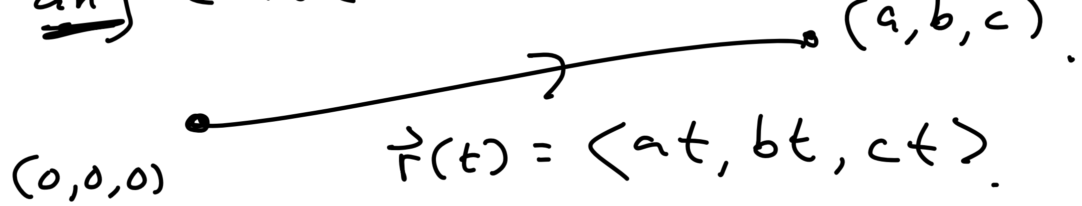
$$\vec{F}(x,y,z) = \nabla f\,(x,y,z).$$

To find it we will use the Fundamental Theorem:

$$\int_{\text{curve}} \vec{F} = \int \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= f(\text{end}) - f(\text{start}).$$

Pick <u>any</u> curve



$\vec{r}(t) = \langle at, bt, ct \rangle.$

$$\int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int \langle bt + ct,\ at + ct,\ at + bt \rangle \cdot \langle a, b, c \rangle\, dt$$

$$= \int_0^1 \cancel{abt} + \cancel{act} + \cancel{abt} + \cancel{bct} + \cancel{act} + \cancel{bct}\ dt$$

$$= 2 \int_0^1 (ab + ac + bc)\, t\, dt$$

$$= ab + ac + bc.$$

Guess: $f(a, b, c) = ab + ac + bc$

$f(x, y, z) = xy + xz + yz$. + any constant

Check:

$f_x = y + z$ ✓

$f_y = x + z$ ✓

$f_z = x + y$ ✓

Problem 5: Div, Grad, Curl.

$f : \mathbb{R}^3 \to \mathbb{R}$
$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

$\nabla f = \langle f_x, f_y, f_z \rangle$

$\nabla \times \vec{F} = \langle \text{- - - -} \rangle$

$\nabla \cdot \vec{F} = P_x + Q_y + R_z$

These 3 "derivatives" satisfy lots of algebraic identities. e.g.

$$\nabla \times (\nabla f) = \langle 0, 0, 0 \rangle$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle P, Q, R \rangle.$$

$$\nabla \times (\nabla f) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle.$$

mixed partials commute
for any reasonable function f

$$= \langle 0, 0, 0 \rangle.$$

Makes Sense: This is part of the theorem that

$$\vec{F} = \nabla f \iff \nabla \times \vec{F} = \langle 0, 0, 0 \rangle.$$

$\Longrightarrow$ ✓

$\Longleftarrow$? Stokes' Theorem

Another algebraic identity:

$$\nabla \cdot (\nabla \times \vec{F}) = 0 \quad \text{(scalar)}$$

$$\nabla \times \vec{F} = \langle \underbrace{R_y - Q_z}_{S}, \underbrace{P_z - R_x}_{T}, \underbrace{Q_x - P_y}_{U} \rangle$$

$$\nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot \langle S, T, U \rangle.$$

$$= S_x + T_y + U_z$$

$$= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}}$$

Again: Mixed partials commute!

$$= 0.$$