HW 5 due Tues.
(Note: New Problem!)

Recall the concept of a conservative vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ in n-dimensional space. Write

$$\vec{F}(x_1,\dots,x_n) = \langle F_1(x_1,\dots,x_n), F_2(x_1,\dots,x_n), \vdots F_n(x_1,\dots,x_n) \rangle$$

The following statements are equivalent:

① $\vec{F} = \nabla f$ for some $f(x_1,\dots,x_n)$
(i.e., $F_i(x_1,\dots,x_n) = f_{x_i}(x_1,\dots,x_n)$)

② $\oint_C \vec{F} = 0$ for any loop C.

[$\oint$ means integral around closed loop] i.e. $\displaystyle\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = 0$

(3) Cross-Partial Property:

$$\frac{dF_i}{dx_j} = \frac{dF_j}{dx_i} \quad \text{for all } i \neq j.$$

A vector field satisfying these conditions is called "conservative".

Main Example:
Gravitational field, say force of gravity on a planet due to the sun. On HW 3 we saw

$$\vec{F}(\vec{r}(t)) = -\frac{\overset{\text{assume } 1}{\cancel{GMm}}}{\|\vec{r}(t)\|^3}\,\vec{r}(t)$$

$$\vec{F}(x,y,z) = -\frac{1}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$

$$= \langle P, Q, R\rangle.$$

So $P(x,y,z) = -x / (x^2+y^2+z^2)^{3/2}$

One can check that cross-partial property is satisfied:

$$P_y = Q_x \quad \& \quad P_z = R_x \quad \& \quad Q_z = R_y.$$

Furthermore, one can show that

$$\vec{F}(x,y,z) = -\nabla f$$

where $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

is called the gravitational potential.

Today we'll focus on 3D case:

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$
$$\vec{F}(x_1,x_2,x_3) = \langle F_1(x_1,x_2,x_3), F_2(x_1,x_2,x_3), F_3(x_1,x_2,x_3) \rangle$$

Consider the "nabla operator"

$$\nabla = \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle = \left\langle \frac{d}{dx_1}, \frac{d}{dx_2}, \frac{d}{dx_3} \right\rangle$$

[Warning: Not really a vector.]

Recall the gradient

$$\nabla f = \underbrace{\left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle}_{\text{"vector"}} \underbrace{f}_{\text{scalar}}$$

CUTE

$$= \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle$$

Use a similar mnemonic to define the "curl" of $\vec{F}$:

$$\nabla \times \vec{F} = \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle \times \langle P, Q, R \rangle$$

$$= \left\langle \frac{dR}{dy} - \frac{dQ}{dz}, \frac{dP}{dz} - \frac{dR}{dx}, \frac{dQ}{dx} - \frac{dP}{dy} \right\rangle$$

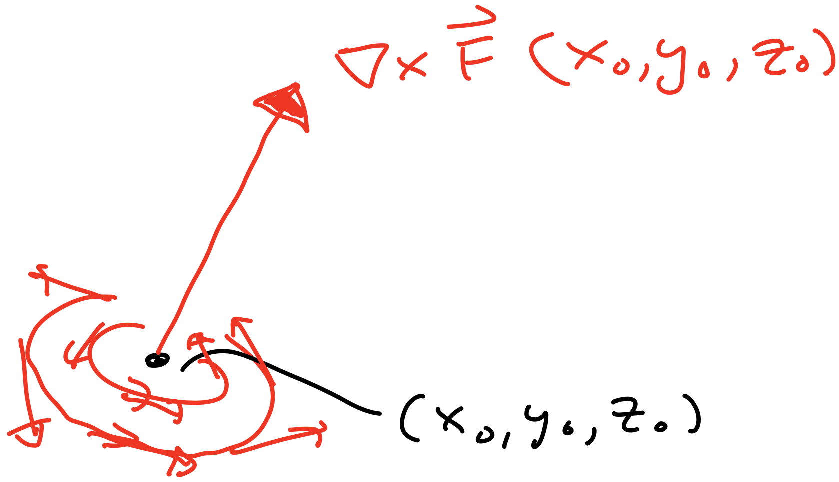
$$= \underbrace{\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle}_{\text{this is a vector field}}$$

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \rightsquigarrow \nabla \times \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

vector field $\rightsquigarrow$ another vector field.

What does it mean?

It represents the amount & the direction of "rotation" in the vector field $\vec{F}$:

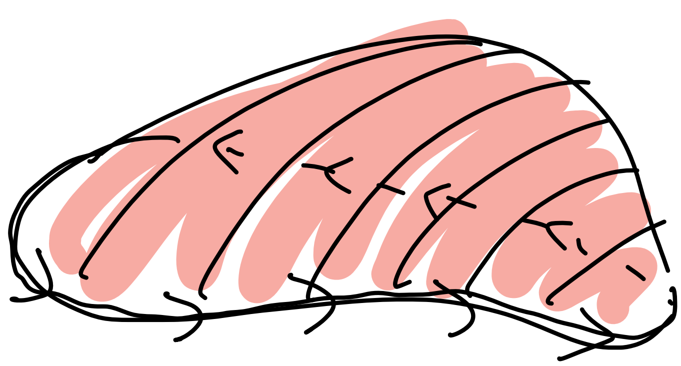


This intuition is based on a Theorem, called <u>Stokes' Theorem</u>.

$$\iint \nabla \times \vec{F} = \int \vec{F}$$

$\iint \nabla \times \vec{F}$: 2D surface in 3D

$\int \vec{F}$: boundary curve of the surface

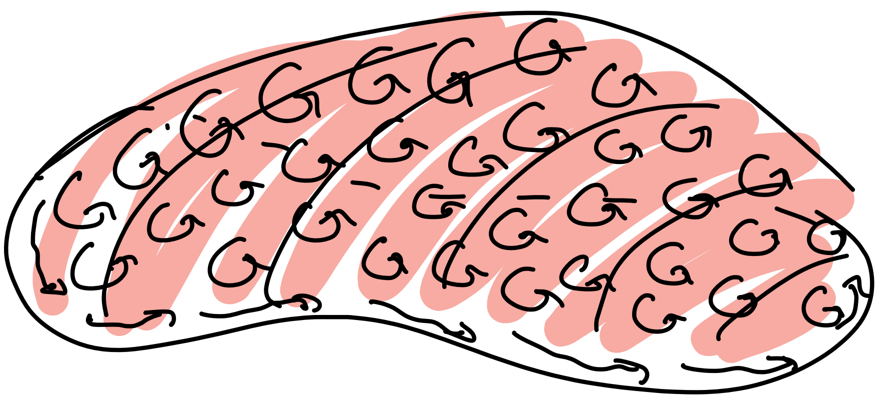


Boundary curve is oriented so surface is "to the left"

TWO QUESTIONS :

- How to define the integral of $\nabla \times \vec{F}$ over a surface?
- Why is it true?

Fake Proof :



All the little rotations cancel, except at the boundary.

How to define $\iint \nabla \times \vec{F}$?

More generally we will define

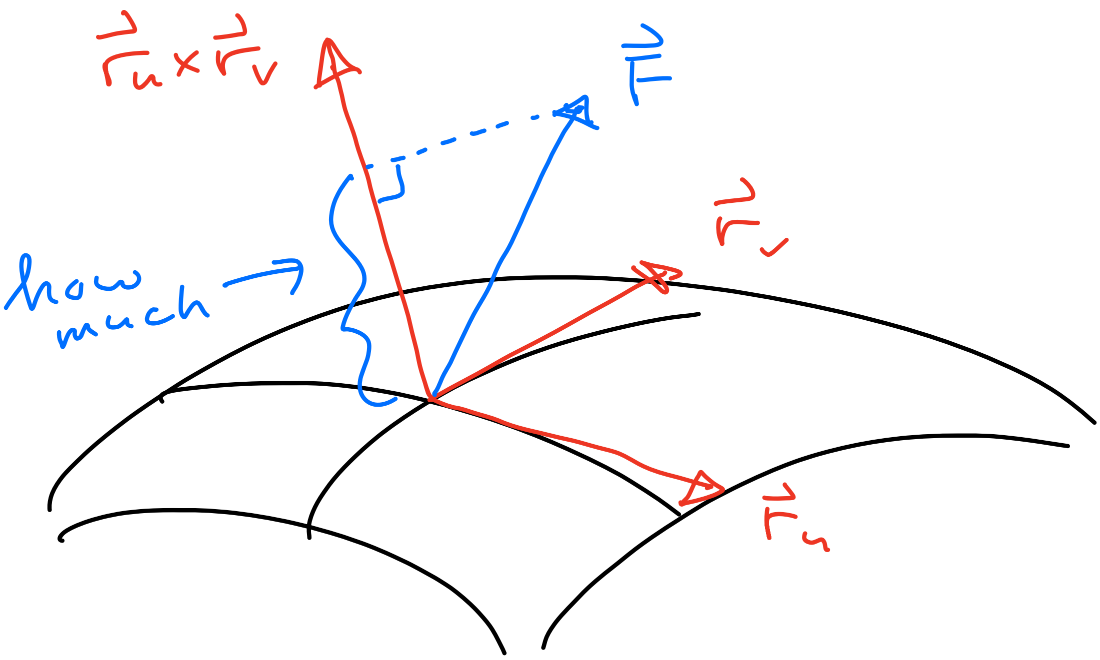
$$\iint_{\text{surface in 3D}} \vec{F}$$

vector field.

Recall: We already know how to integrate a scalar over a surface

$$\iint f(\vec{r}(u,v)) \underbrace{\|\vec{r}_u \times \vec{r}_v\| \, du\, dv}_{\text{tiny area}}$$

To define the integral of a vector field $\vec{F}$ we will integrate the "normal component of $\vec{F}$", which is a scalar measuring the amount of $\vec{F}$ $\perp$ to the surface.



$$\text{how much} = \frac{\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)}{\|\vec{r}_u \times \vec{r}_v\|}$$

So the definition of the integral is

$$\iint_{\text{surface}} \vec{F}$$

$$= \iint \underbrace{\frac{\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)}{\|\vec{r}_u \times \vec{r}_v\|}}_{\text{scalar}} \underbrace{\|\vec{r}_u \times \vec{r}_v\| \, du\, dv}_{\text{tiny area}}$$

Some intuition:

$\vec{F}$ = velocity field of a fluid (of constant mass density)

Then $\iint_{\text{surface}} \vec{F}$ = rate of flow across the surface

= volume (mass) / area / time.

This is why $\iint_{\text{surface}} \vec{F}$ is often called a "flux" integral

"flux" = "flow"

In the case of Stokes Thm, we don't think of $\nabla \times \vec{F}$ as "velocity", but the math definition is the same,

To be precise:

$\vec{r}(u,v)$ = parametrized surface

$\vec{r}(t)$ = parametrized boundary curve.

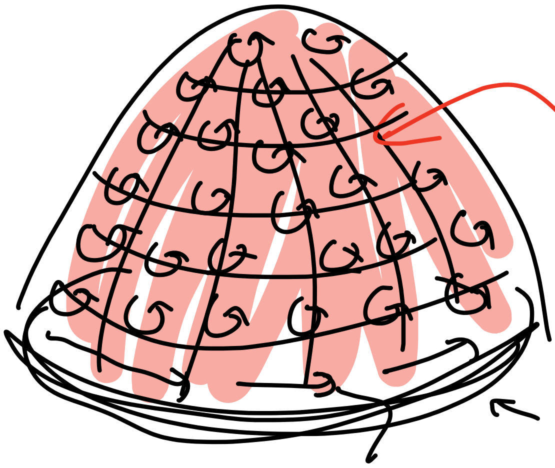
$$\iint_{\text{surface}} \nabla \times \vec{F} = \int_{\text{curve}} \vec{F}$$

$$\iint (\nabla \times \vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Example: Check Stokes' Thm for $\vec{F}(x,y,z) = \langle -2y, 2x, x^2 z \rangle$ over the surface $z = 1 - x^2 - y^2$ for $z \geq 0$:

$u^2 = x^2 + y^2$



parabolic dome

$\vec{r}(u,v) =$

$\langle u\cos v, u\sin v, 1-u^2 \rangle$

$\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$

Summary:

$\nabla \times \vec{F} = \langle 0, -2xz, 4 \rangle$

$\nabla \times \vec{F}(\vec{r}(u,v)) = \langle 0, -2u\cos v(1-u^2), 4 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u^2\cos v, 2u^2\sin v, u \rangle$

$$\iint (\nabla \times \vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= \int_{v=0}^{2\pi}\int_{u=0}^{1} \left(4u - 4u^3\cos v(1-u^2)\right) du\, dv$$

$= \underset{\text{computer}}{\cdots\cdots} = 4\pi$

Now integrate along boundary curve:

$\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$

$\vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$

$\vec{F}(\vec{r}(t)) = \langle -2y, 2x, x^2 z \rangle$

$= \langle -2\sin t, 2\cos t, 0 \rangle$

$\int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$

$= \int \langle -2\sin t, 2\cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt$

$= \int (+2\sin^2 t + 2\cos^2 t)\, dt$

$= \int_0^{2\pi} 2\, dt = 4\pi$ ✓

This way is easier!

Why do we care?

Suppose $\nabla \times \vec{F} = \langle 0, 0, 0 \rangle$

i.e. suppose

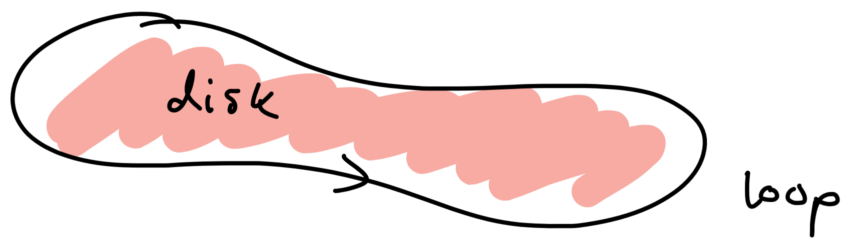
$$\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle 0,0,0 \rangle$$

i.e. suppose $\begin{cases} R_y = Q_z \\ P_z = R_x \\ Q_x = P_y \end{cases}$

Then from Stokes' Theorem we get

$$\oint_{\text{loop}} \vec{F} = 0.$$

Indeed:

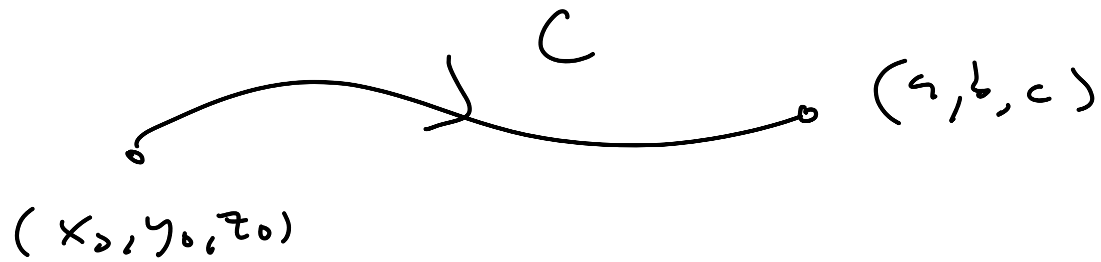


$$\oint_{\text{loop}} \vec{F} = \iint_{\text{disk}} \nabla \times \vec{F}$$

$$= \iint_{\text{disk}} \langle 0,0,0 \rangle = 0.$$

Then we can show that $\vec{F}$ has an antiderivative: $\vec{F} = \nabla f$.

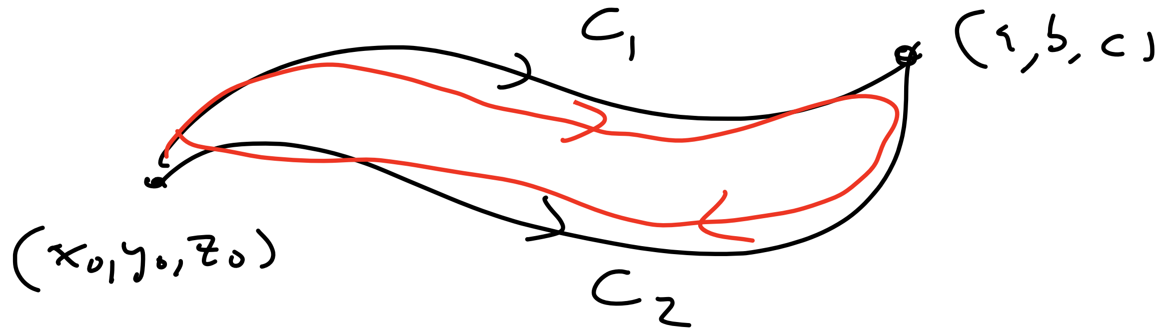
Idea: Want to define $f(a,b,c)$ for any point (a,b,c). Fix a basepoint (x_0,y_0,z_0) and choose any curve



Define $f(a,b,c) = \int_C \vec{F}$

Then it will follow from Fund Thm of Line integrals that $\nabla f = \vec{F}$.

Only one problem. How do we know that the curve C is arbitrary? Take two curves:



Consider the loop "$C_1 - C_2$"

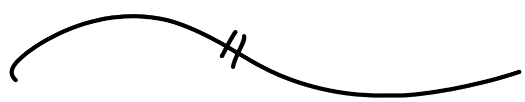
Then

$$\int_{C_1} \vec{F} - \int_{C_2} \vec{F} = \int_{C_1 - C_2} \vec{F} = 0$$

this is a loop

$$\int_{C_1} \vec{F} = \int_{C_2} \vec{F}$$

so the path really doesn't matter.



That was fancy. Let's go to 2D.

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle.$$

$$\nabla \times \vec{F} = \ ?$$

Makes no sense because cross product is a 3D concept.

But we can fake it:

$\vec{F} = \langle P, Q, 0 \rangle$ nothing happens in the z-direction.

$\nabla \times \vec{F} = \langle 0, 0, \underbrace{Q_x - P_y}_{\text{this tells you about rotation in the xy-plane.}} \rangle$

Define the 2D curl:

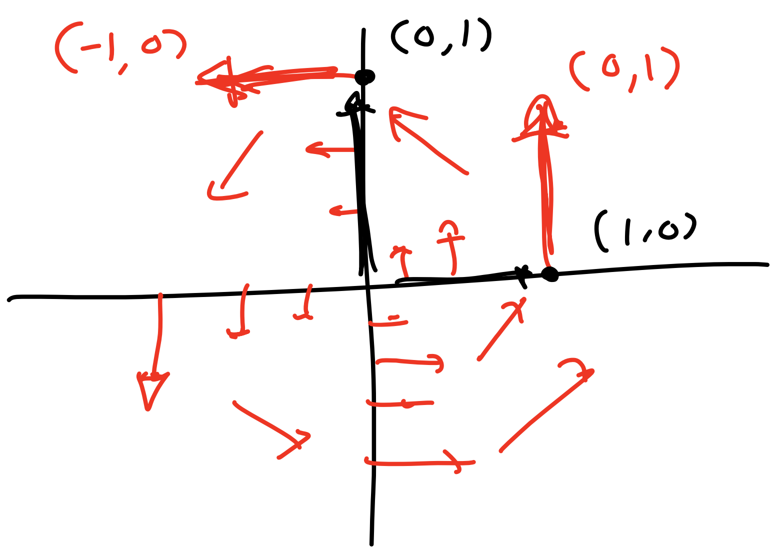
$$\text{curl}(\vec{F}) = Q_x - P_y$$

$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \rightsquigarrow \text{curl}(\vec{F}) : \mathbb{R}^2 \to \mathbb{R}$

vector field $\longrightarrow$ scalar field.

Typical Example:

$$\vec{F} = \langle -y, x \rangle = \langle P, Q \rangle$$

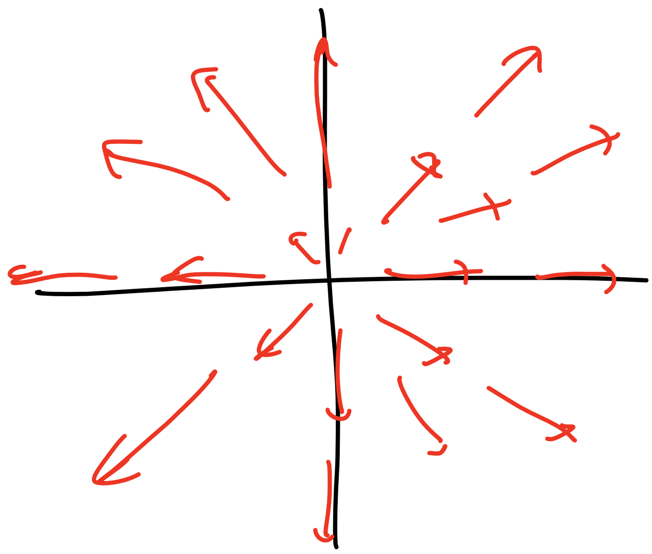


curl should be nonzero, because this field is rotating.

$$\mathrm{curl}(\vec{F}) = Q_x - P_y$$
$$= 1 - (-1) = 2 > 0$$

Indicates counterclockwise rotation.

Now consider $\vec{G} = \langle x, y \rangle = \langle P, Q \rangle$



$$\mathrm{curl}(\vec{G}) = Q_x - P_y = 0 - 0 = 0$$

This field is not rotating.

Preview of the Bonus Lecture:

$$\operatorname{div}(\vec{F}) = P_x + Q_y.$$

$$\operatorname{div}\langle -y, x\rangle = 0$$

$$\operatorname{div}\langle x, y\rangle = 2$$

This measures expansion.