Problem 1. Lines in $\mathbb{R}^3$.

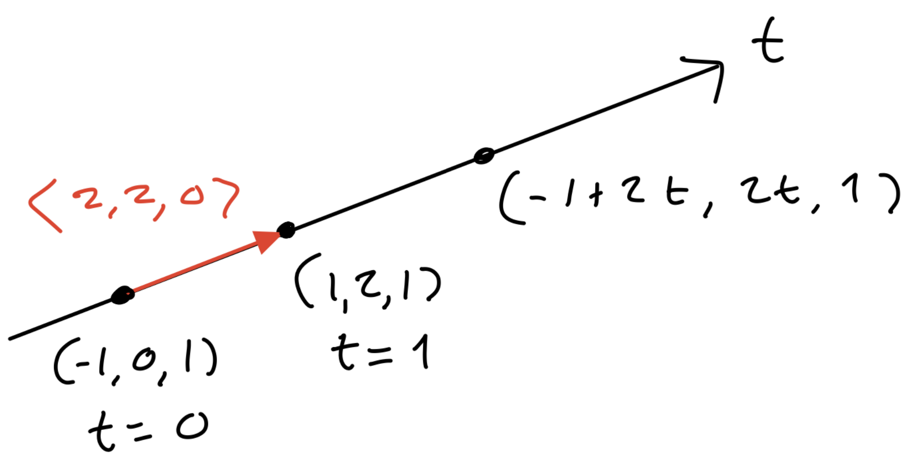
(a) Find a parametrization for the line passing through $P = (-1, 0, 1)$ and $Q = (1, 2, 1)$.
(b) Find a parametrization for the line of intersection of the following two planes:

$$\begin{matrix} (1) \\ (2) \end{matrix} \left\{ \begin{array}{ccccccc} x & + & 2y & + & 2z & = & 1, \\ x & + & 3y & + & 5z & = & 3. \end{array} \right.$$

(a): We need one point on the line and one vector in the line. We will choose the point $P = (-1, 0, 1)$ and the vector $\vec{PQ} = \langle 1 - (-1), 2 - 0, 1 - 1 \rangle = \langle 2, 2, 0 \rangle$. Then every point of the line has the form $\mathbf{r}(t)$ where

$$\mathbf{r}(t) = \langle -1, 0, 1 \rangle + r\langle 2, 2, 0 \rangle = \langle -1 + 2t, 2t, 1 \rangle.$$

Here is a picture:



(b): First we subtract (1) from (2) to obtain an equation with no x:

$$(3) = (2) - (1) : 0 + y + 3z = 2.$$

Then we subtract 2(3) from (1) to obtain an equation with no y:

$$(4) = (1) - 2(3) : x + 0 - 4z = -3.$$

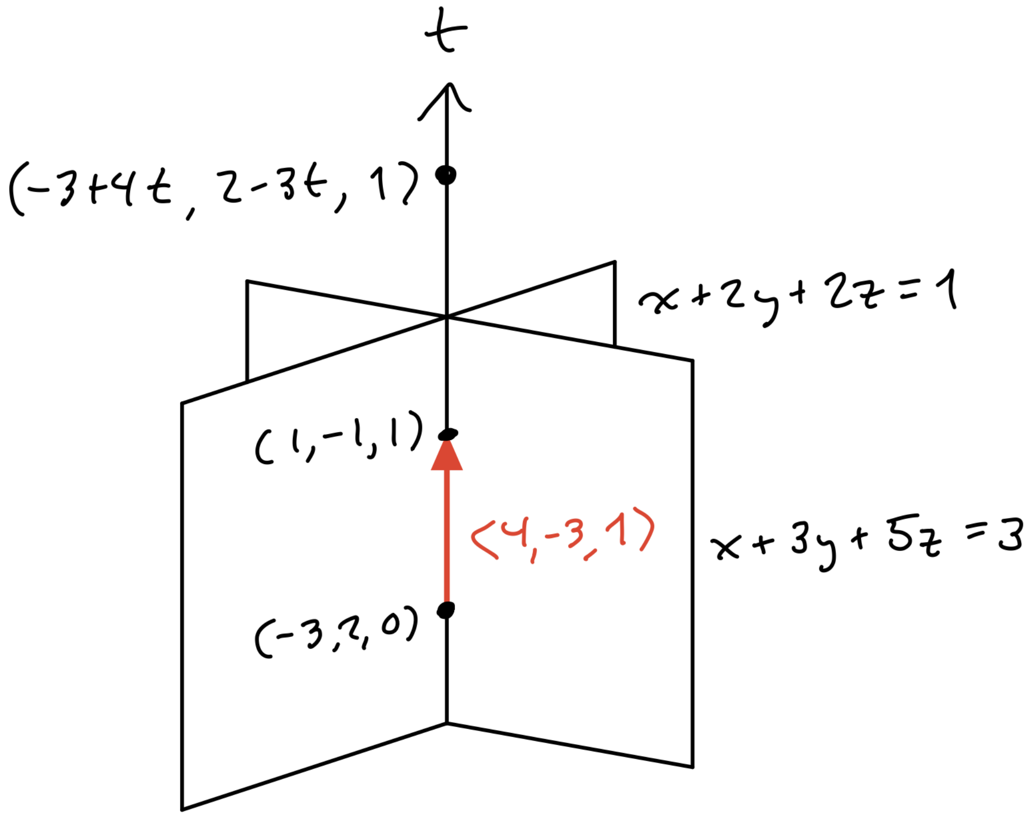
Finally, we let $t = z$ be a parameter and solve for (x, y, z) in terms of t:

$$\left\{ \begin{array}{ll} x & = -3 + 4t, \\ y & = 2 - 3t, \\ z & = t. \end{array} \right.$$

We can also express this as

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle -3 + 4t, 2 - 3t, t \rangle = \langle -3, 2, 0 \rangle + t\langle 4, -3, 1 \rangle.$$

Here is a picture:



Problem 2. Integration of Vector-Valued Functions. Consider a parametrized curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ with the following properties:

$$\begin{aligned}\mathbf{r}(0) &= \langle 2, 1\rangle,\\ \mathbf{r}'(0) &= \langle 0, 1\rangle,\\ \mathbf{r}''(t) &= \langle -\cos t, -\sin t\rangle.\end{aligned}$$

(a) Integrate $\mathbf{r}''(t)$ to obtain $\mathbf{r}'(t)$.
(b) Integrate $\mathbf{r}'(t)$ to obtain $\mathbf{r}(t)$.

(a): We have

$$\begin{aligned}\mathbf{r}'(t) = \int \mathbf{r}''(t)\,dt &= \left\langle \int (-\cos t)\,dt, \int (-\sin t)\,dt \right\rangle\\ &= \langle -\sin t + c_1, \cos t + c_2\rangle,\end{aligned}$$

for some constants c_1, c_2. To find these constants, we evaluate at 0:

$$\langle 0, 1\rangle = \mathbf{r}'(0) = \langle -\sin 0 + c_1, \cos 0 + c_2\rangle = \langle c_1, 1 + c_2\rangle.$$

We find that $c_1 = 0$ and $c_2 = 0$, so that

$$\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle.$$

(b): Then we integrate again to get

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \left\langle \int (-\sin t)\,dt, \int (\cos t)\,dt \right\rangle$$
$$= \langle \cos t + c_3, \sin t + c_4 \rangle.$$

To find the constants c_3, c_4 we evaluate at $t = 0$:

$$\langle 2, 1 \rangle = \mathbf{r}(0) = \langle \cos 0 + c_3, \sin 0 + c_4 \rangle = 1 + c_3, 0 + c_4 \rangle.$$

This implies that $c_3 = 1$ and $c_4 = 1$, hence

$$\mathbf{r}(t) = \langle 1 + \cos t, 1 + \sin t \rangle.$$

Remark: I created this problem by starting with a parametrized circle with center at $\langle 1, 1 \rangle$ and radius 1. Here is a picture: