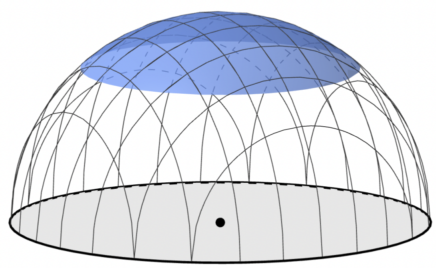
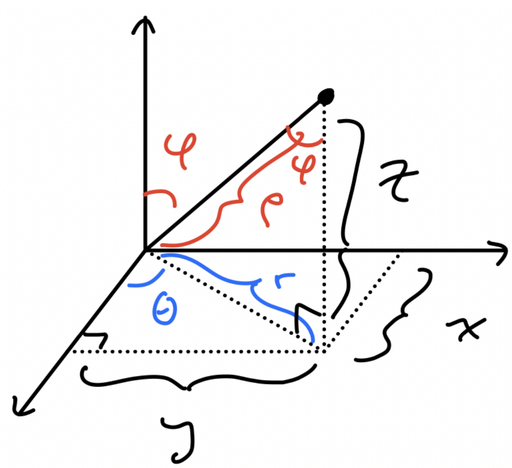
Math 211 **Summer 2022**
Homework 5 **Drew Armstrong**

Problem 1. Surface Area. Fix an angle $0 \leq \alpha < \pi$ and let D be the region on the surface of a sphere of radius 1 with angle $\leq \alpha$ from the vertical:[1]



(a) Find a parametrization for D of the form $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$.
(b) Use your parametrization to compute the surface area of D.

(a): We will use spherical coordinates (ρ, θ, φ) with a fixed radius $\rho = 1$. Recall that spherical coordinates are connected to polar and Cartesian coordinates via two right triangles:[2]



From the picture we obtain

$$\left\{ \begin{array}{ll} x & = r\cos\theta \\ y & = r\sin\theta \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{ll} z & = \rho\cos\varphi \\ r & = \rho\sin\varphi \end{array} \right\}, \quad \text{hence} \quad \left\{ \begin{array}{ll} x & = \rho\sin\varphi\cos\theta \\ y & = \rho\sin\varphi\sin\theta \\ z & = \rho\cos\varphi \end{array} \right\}$$

After fixing $\rho = 1$ this becomes

$$\begin{aligned} \mathbf{r}(\theta, \varphi) &= \langle x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi)\rangle \\ &= \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle \end{aligned}$$

This parametrization covers the whole surface of the unit sphere as $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$. In this problem we are only interested in the region where $0 \leq \varphi \leq \alpha$, for the fixed angle α.

[1] On the Earth, this is the region above latitute $(90 - \alpha)$ degrees North.

[2] Different books use different naming conventions. Instead of memorizing the formulas, just memorize the picture. Then you can derive the formulas for yourself.

(b): To compute the surface area, we first need the stretch factor $\|\mathbf{r}_\theta \times \mathbf{r}_\varphi\|$. We have

$$\begin{aligned}
\mathbf{r}_\theta &= \langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\rangle,\\
\mathbf{r}_\varphi &= \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\rangle,\\
\mathbf{r}_\theta \times \mathbf{r}_\varphi &= \langle -\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi\sin^2\theta - \sin\varphi\cos\varphi\cos^2\theta\rangle\\
&= \langle -\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi\rangle,
\end{aligned}$$

and hence

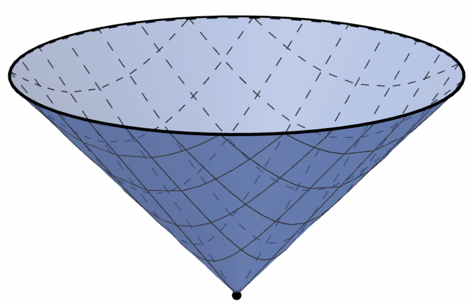
$$\begin{aligned}
\|\mathbf{r}_\theta \times \mathbf{r}_\varphi\|^2 &= \sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi\\
&= \sin^4\varphi + \sin^2\varphi\cos^2\varphi\\
&= \sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)\\
&= \sin^2\varphi,\\
\|\mathbf{r}_\theta \times \mathbf{r}_\varphi\| &= \sin\varphi.
\end{aligned}$$

We don't need to write $|\sin\varphi|$ because in spherical coordinates we always have $0 \le \varphi \le \pi$. Finally, we compute the area:

$$\begin{aligned}
\iiint_D 1\,dA &= \iint_D \|\mathbf{r}_\theta \times \mathbf{r}_\varphi\|\,d\theta d\varphi\\
&= \int_0^{2\pi} d\theta \cdot \int_0^{\alpha} \sin\varphi\,d\varphi\\
&= 2\pi\left(-\cos(\alpha) + \cos(0)\right)\\
&= 2\pi(1-\cos\alpha).
\end{aligned}$$

Check: When $\alpha = 0$ we have area 0, as exected. When $\alpha = \pi/2$ we have area 2π which is the correct area of the hemisphere. When $\alpha = \pi$ we have 4π which is the correct surface area of the full unit sphere.

Problem 2. Surface Area. Let D be the surface of the cone $z^2 = x^2 + y^2$ for values z between 0 and 1:



(a) Find a parametrization for D of the form $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$.
(b) Use your parametrization to compute the surface area of D.

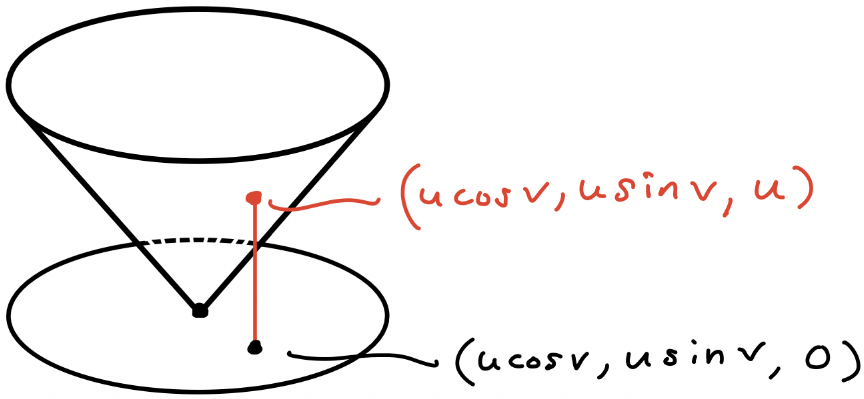
(a): We will use polar coordinates $x = u\cos v$ and $y = u\sin v$.[3] Then the equation of the cone becomes $z^2 = x^2 + y^2 = u^2$, or $z = u$ (since z and u are both positive). As z goes from 0 to

[3] I don't write $x = r\cos\theta$ and $y = r\sin\theta$ because we are already using the letter $\mathbf{r}$.

1, so does u. Hence the surface of the cone has the following parametrization:

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u\rangle \qquad \text{where } 0 \le u \le 1 \text{ and } 0 \le v \le 2\pi.$$

Here is a picture:



(b): First we compute the stretch factor:

$$\begin{aligned}
\mathbf{r}_u &= \langle \cos v, \sin v, 1\rangle, \\
\mathbf{r}_v &= \langle -u\sin v, u\cos v, 0\rangle, \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle -u\cos v, -u\sin v, -u\sin^2 v - u\cos^2 v\rangle \\
&= \langle -u\cos v, -u\sin v, -u\rangle, \\
\|\mathbf{r}_u \times \mathbf{r}_v\|^2 &= u^2\cos^2 v + u^2\sin^2 v + u^2 \\
&= u^2 + u^2 \\
&= 2u^2, \\
\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{2}\cdot u.
\end{aligned}$$

Then we compute the area:

$$\begin{aligned}
\iiint_D 1\, dA = \iiint_D \sqrt{2}\cdot u\, du dv \\
&= \sqrt{2}\cdot \int_0^{2\pi} dv \cdot \int_0^1 u\, du \\
&= \sqrt{2}\cdot 2\pi \cdot (1/2) \\
&= \sqrt{2}\cdot \pi.
\end{aligned}$$

Remark: More generally, the surface area of a cone with height h and base a circle of radius a is $\pi a\sqrt{h^2 + a^2}$. In our case we have $a = h = 1$.

Problem 3. Gravitational Potential Near the Surface of a Planet. Choose a coordinate system near the surface of a planet, so that $z = 0$ is the ground and the z-axis points "up". A particle of mass m at a point (x, y, z) with $z \ge 0$ feels a constant gravitational force of $\mathbf{F}(x, y, z) = \langle 0, 0, -mg\rangle$.

(a) Suppose that the particle has initial position and initial velocity as follows:

$$\mathbf{r}(0) = \langle 0, 0, 0\rangle,$$
$$\mathbf{r}'(0) = \langle u, v, w\rangle.$$

Integrate Newton's equation $\mathbf{F} = m\mathbf{r}''(t)$ to find $\mathbf{r}'(t)$ and $\mathbf{r}(t)$.

(b) Find a formula for the kinetic energy at time t:

$$\mathrm{KE}(t) = \frac{1}{2}m\|\mathbf{r}'(t)\|^2.$$

(c) Find a scalar field $f(x, y, z)$ such that $\mathbf{F} = -\nabla f$ and $f(0, 0, 0) = 0$. This f is called the *gravitational potential* of the particle.[4]

(d) Find a formula for the potential energy at time t:

$$\mathrm{PE}(t) = f(\mathbf{r}(t)).$$

(e) Check that the total mechanical energy $\mathrm{KE}(t) + \mathrm{PE}(t)$ is constant.

(a): Since $\mathbf{F} = \langle 0, 0, -mg\rangle$, Newton's 2nd Law tells us that

$$\begin{aligned} m\mathbf{r}''(t) &= \mathbf{F} \\ m\mathbf{r}''(t) &= \langle 0, 0, -mg\rangle \\ \mathbf{r}''(t) &= \langle 0, 0, -g\rangle. \end{aligned}$$

In other words, the particle has constant acceleration. We integrate this once to get

$$\mathbf{r}'(t) = \langle c_1, c_2, -gt + c_3\rangle,$$

for some constants c_1, c_2, c_3. The initial condition $\mathbf{r}'(0) = \langle u, v, w\rangle$ tells us that $\langle c_1, c_2, c_3\rangle = \langle u, v, w\rangle$, so the velocity at time t is

$$\mathbf{r}'(t) = \langle u, v, -gt + w\rangle.$$

We integrate again to obtain

$$\mathbf{r}(t) = \left\langle ut + c_3, vt + c_4, -\frac{1}{2}gt^2 + wt + c_6\right\rangle$$

for some constants c_4, c_5, c_6. Then the initial condition $\mathbf{r}(0) = \langle 0, 0, 0\rangle$ tells us that $\langle c_4, c_5, c_6\rangle = \langle 0, 0, 0\rangle$, so the position at time t is

$$\mathbf{r}(t) = \left\langle ut, vt, -\frac{1}{2}gt^2 + wt\right\rangle.$$

(b): The kinetic energy at time t is

$$\begin{aligned} \mathrm{KE}(t) &= \frac{1}{2}m\|\mathbf{r}'(t)\|^2 \\ &= \frac{1}{2}m\|\langle u, v, -gt + w\rangle\|^2 \\ &= \frac{1}{2}m\left(u^2 + v^2 + (-gt + w)^2\right) \\ &= \frac{1}{2}m\left(u^2 + v^2 + g^2t^2 - 2gtw + w^2\right) \\ &= \frac{1}{2}m(u^2 + v^2 + w^2) + \frac{1}{2}mg^2t^2 - mgtw. \end{aligned}$$

[4] Actually, the choice $f(0, 0, 0) = 0$ is arbitrary. We are just saying that a particle on the ground has zero gravitational potential. Only **changes** in potential energy are physically meaningful.

(c): A constant vector field is necessarily conservative. For example, consider $\mathbf{F} = \langle a, b, c\rangle$ for some constants a, b, c. Then we observe that $\mathbf{F} = \nabla f$ where $f(x, y, z) = ax + by + cz$. Indeed, it is easy to check that $\nabla f = \langle f_x, f_y, f_z\rangle = \langle a, b, c\rangle$. We could find this f by brute force, or we could use the Fundamental Theorem of Line Integrals. For a given point (x, y, z) we will integrate $\mathbf{F}$ along the path $\mathbf{r}(t) = \langle xt, yt, zt\rangle$ for t from 0 to 1. If $\mathbf{F} = \nabla f$ then we must have

$$\begin{aligned} f(x,y,z) - f(0,0,0) &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\ &= \int_0^1 \mathbf{F}(xt, yt, zt) \bullet \langle x, y, z\rangle\, dt \\ &= \int_0^1 \langle a, b, c\rangle \bullet \langle x, y, z\rangle\, dt \\ &= \int_0^1 (ax + by + cz) t\, dt \\ &= ax + by + cz. \end{aligned}$$

In the case when $\mathbf{F} = \langle a, b, c\rangle = \langle 0, 0, -mg\rangle$ we have $\mathbf{F} = \nabla f$ where $f(x, y, z) = 0x+0y-mgz = -mgz$. But for physical reasons we write $\mathbf{F} = -\nabla f$ with $f(x, y, z) = -mgz$.

(d): The potential energy at time t is

$$\begin{aligned} \mathrm{PE}(t) &= f(\mathbf{r}(t)) \\ &= f\left(ut, vt, -\frac{1}{2}gt^2 + wt\right) \\ &= mg\left(-\frac{1}{2}gt^2 + wt\right) \\ &= -\frac{1}{2}mg^2t^2 + mgwt. \end{aligned}$$

(e): From parts (b) and (d) we see that

$$\mathrm{KE}(t) + \mathrm{PE}(t) = \frac{1}{2}m(u^2 + v^2 + w^2),$$

which is independent of t.

Problem 4. Conservative Vector Fields. Consider the following vector fields:

$$\begin{aligned} \mathbf{F}(x,y,z) &= \langle y+z, x+z, x+y\rangle, \\ \mathbf{G}(x,y,z) &= \langle -y+z, x+z, x+y\rangle. \end{aligned}$$

(a) Compute $\nabla \times \mathbf{F}$ and $\nabla \times \mathbf{G}$. Observe that $\mathbf{F}$ is conservative, while $\mathbf{G}$ is not.
(b) Now think of $\mathbf{F}$ and $\mathbf{G}$ as force fields. Compute the work done by $\mathbf{F}$ and $\mathbf{G}$ on a particle of mass 1 traveling around the circle $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$ for $0 \le t \le 2\pi$.
(c) Find a scalar field $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

(a): We have

$$\begin{aligned} \nabla \times \mathbf{F} &= \langle (x+y)_y - (x+z)_z, (y+z)_z - (x+y)_x, (x+z)_x - (y+z)_y\rangle \\ &= \langle 1-1, 1-1, 1-1\rangle \\ &= \langle 0,0,0\rangle \end{aligned}$$

and

$$\begin{aligned}\nabla \times \mathbf{G} &= \langle (x+y)_y - (x+z)_z, (-y+z)_z - (x+y)_x, (x+z)_x - (-y+z)_y \rangle \\ &= \langle 1-1, 1-1, 1-(-1) \rangle \\ &= \langle 0, 0, 2 \rangle.\end{aligned}$$

This tells us that $\mathbf{F}$ is conservative, while $\mathbf{G}$ is not.

(b): The work done by a force field $\mathbf{F}$ acting on moving particle $\mathbf{r}(t)$ is defined as

$$\int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt.$$

In the case of $\mathbf{F}(x,y,z) = \langle y+z, x+z, x+y \rangle$ and $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$ we have

$$\begin{aligned}\int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt &= \int \mathbf{F}(\cos t, \sin t, 0) \bullet \langle -\sin t, \cos t, 0 \rangle\,dt \\ &= \int \langle \sin t + 0, \cos t + 0, \cos t + \sin t \rangle \bullet \langle -\sin t, \cos t, 0 \rangle\,dt \\ &= \int (-\sin^2 t + \cos^2 t + 0)\,dt \\ &= \int \cos(2t)\,dt \\ &= \left[\frac{1}{2}\sin(2t)\right]_0^{2\pi} \\ &= \frac{1}{2}\sin(4\pi) - \frac{1}{2}\sin(0) \\ &= 0 - 0 \\ &= 0.\end{aligned}$$

This was expected because the integral of a conservative vector around any loop is zero. In the case of $\mathbf{G} = \langle -y+z, x+z, x+y \rangle$ we have

$$\begin{aligned}\int \mathbf{G}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt &= \int \mathbf{F}(\cos t, \sin t, 0) \bullet \langle -\sin t, \cos t, 0 \rangle\,dt \\ &= \int \langle -\sin t + 0, \cos t + 0, \cos t + \sin t \rangle \bullet \langle -\sin t, \cos t, 0 \rangle\,dt \\ &= \int (\sin^2 t + \cos^2 t + 0)\,dt \\ &= \int_0^{2\pi} 1\,dt \\ &= 2\pi.\end{aligned}$$

The fact that this integral is not zero again verifies that the vector field $\mathbf{G}$ is not conservative.

(c): We are looking for a scalar field $f(x,y,z)$ satisfying

$$\langle f_x, f_y, f_z \rangle = \langle y+z, x+z, x+y \rangle.$$

We will do this in two ways.

Brute Force. Since $f_x(x,y,z) = y+z$ we must have

$$f(x,y,z) = xy + xz + g(y,z) \text{ for some function } g(y,z).$$

Then since $f(x,y,z) = xy + xz + g(y,z)$ and $f_y(x,y,z) = x + z$ we must have

$$\begin{aligned}
x + g_y(y,z) &= x + z \\
g_y(y,z) &= z \\
g(y,z) &= yz + h(z) \text{ for some function } h(z).
\end{aligned}$$

Finally, since $f(x,y,z) = xy + xz + yz + h(z)$ and $f_z(x,y,z) = x + y$ we must have

$$\begin{aligned}
x + y + h_z(z) &= x + y \\
h_z(z) &= 0 \\
h(z) &= \text{constant}.
\end{aligned}$$

We conclude that $f(x,y,z) = xy + xz + yz$, plus some arbitrary constant.

Use the Fundamental Theorem of Line Integrals. If $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$ then for any path C we have

$$\int_C \mathbf{F} = f(\text{end point of } C) - f(\text{start point of } C).$$

In particular, if we choose the path $\mathbf{r}(t) = \langle xt, yt, zt\rangle$ for $0 \le t \le 1$ then we obtain

$$\begin{aligned}
f(x,y,z) - f(0,0,0) &= \int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int \mathbf{F}(xt, yt, zt) \bullet \langle x, y, z\rangle\, dt \\
&= \int \langle yt + zt, xt + zt, xt + yt\rangle \bullet \langle x, y, z\rangle\, dt \\
&= \int \left((yt + zt)x + (xt + zt)y + (xt + yt)z\right)\, dt \\
&= 2(xy + xz + yz) \cdot \int_0^1 t\, dt \\
&= xy + xz + yz.
\end{aligned}$$

Hence $f(x,y,z) = xy + xz + yz + f(0,0,0)$, where $f(0,0,0)$ is just some arbitrary constant. I like this method better because it doesn't require any cleverness.

Finally, let's check that we got the right answer:

$$\begin{aligned}
\nabla(xy + xz + yz) &= \langle (xy + xz + yz)_x, (xy + xz + yz)_y, (xy + xz + yz)_z\rangle \\
&= \langle y + z + 0, x + 0 + z, 0 + x + y\rangle. \\
&= \mathbf{F}(x,y,z).
\end{aligned}$$

This again confirms that $\mathbf{F}$ is a conservative vector field.[5]

Problem 5. Div, Grad, Curl. Consider a scalar field $f(x,y,z)$ and a vector field $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$. Then we define vector fields called the "gradient of f" and the "curl of $\mathbf{F}$":

$$\text{grad}(f) = \nabla f = \langle f_x, f_y, f_z\rangle,$$

[5] We could try to use these methods to find a scalar function $g(x,y,z)$ such that $\mathbf{G}(x,y,z) = \nabla g(x,y,z)$. The first method completely fails. The second method seems to work, but it spits out $g(x,y,z) = xz + yz$, which is **not** an antiderivative of $\mathbf{G}$. Moral: Always check that the curl is zero before you try to find an antiderivative.

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

We also define a scalar field called the "divergence of $\mathbf{F}$":

$$\operatorname{div}(\mathbf{F}) = \nabla \bullet \mathbf{F} = P_x + Q_y + R_z.$$

(a) Check that $\operatorname{curl}(\operatorname{grad}(f)) = \nabla \times (\nabla f) = \langle 0, 0, 0 \rangle$.
(b) Check that $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = \nabla \bullet (\nabla \times \mathbf{F}) = 0$.

(a): Write $\nabla f = \langle f_x, f_y, f_z \rangle = \langle P, Q, R \rangle$. Then we have

$$\begin{aligned}
\nabla \times (\nabla f) &= \nabla \times \langle P, Q, R \rangle \\
&= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\
&= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle \\
&= \langle 0, 0, 0 \rangle.
\end{aligned}$$

Here we used the fact that mixed partials commute for any reasonable function.[6]

(b): Write $\nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle S, T, U \rangle$. Then we have

$$\begin{aligned}
\nabla \bullet (\nabla \times \mathbf{F}) &= S_x + T_y + U_z \\
&= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z \\
&= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \\
&= (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) + (R_{yx} - R_{xy}) \\
&= 0.
\end{aligned}$$

Here again we used the fact that mixed partials commute.

[6]I guess I should have mentioned that we restrict our attention to reasonable functions.