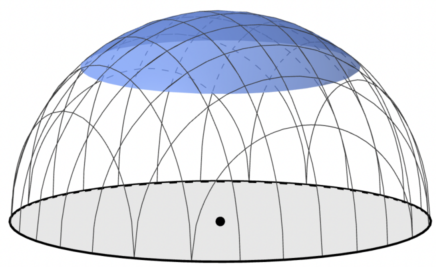
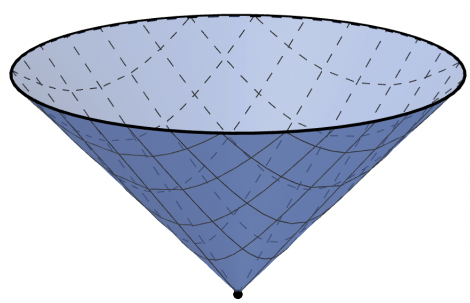
Math 211 **Summer 2022**
Homework 5 **Drew Armstrong**

Problem 1. Surface Area. Fix an angle $0 \leq \alpha < \pi$ and let D be the region on the surface of a sphere of radius 1 with angle $\leq \alpha$ from the vertical:[1]



(a) Find a parametrization for D of the form $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$.
(b) Use your parametrization to compute the surface area of D.

Problem 2. Surface Area. Let D be the surface of the cone $z^2 = x^2 + y^2$ for values z between 0 and 1:



(a) Find a parametrization for D of the form $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$.
(b) Use your parametrization to compute the surface area of D.

Problem 3. Gravitational Potential Near the Surface of a Planet. Choose a coordinate system near the surface of a planet, so that $z = 0$ is the ground and the z-axis points "up". A particle of mass m at a point (x, y, z) with $z \geq 0$ feels a constant gravitational force of $\mathbf{F}(x, y, z) = \langle 0, 0, -mg\rangle$.

(a) Suppose that the particle has initial position and initial velocity as follows:

$$\mathbf{r}(0) = \langle 0, 0, 0\rangle,$$
$$\mathbf{r}'(0) = \langle u, v, w\rangle.$$

Integrate Newton's equation $\mathbf{F} = m\mathbf{r}''(t)$ to find $\mathbf{r}'(t)$ and $\mathbf{r}(t)$.

[1] On the Earth, this is the region above latitute $(90 - \alpha)$ degrees North.

(b) Find a formula for the kinetic energy at time t:

$$\text{KE}(t) = \frac{1}{2}m\|\mathbf{r}'(t)\|^2.$$

(c) Find a scalar field $f(x, y, z)$ such that $\mathbf{F} = -\nabla f$ and $f(0, 0, 0) = 0$. This f is called the *gravitational potential* of the particle.[2]

(d) Find a formula for the potential energy at time t:

$$\text{PE}(t) = f(\mathbf{r}(t)).$$

(e) Check that the total mechanical energy $\text{KE}(t) + \text{PE}(t)$ is constant.

Problem 4. Conservative Vector Fields. Consider the following vector fields:

$$\mathbf{F}(x, y, z) = \langle y + z, x + z, x + y\rangle,$$
$$\mathbf{G}(x, y, z) = \langle -y + z, x + z, x + y\rangle.$$

(a) Compute $\nabla \times \mathbf{F}$ and $\nabla \times \mathbf{G}$. Observe that $\mathbf{F}$ is conservative, while $\mathbf{G}$ is not.

(b) Now think of $\mathbf{F}$ and $\mathbf{G}$ as force fields. Compute the work done by $\mathbf{F}$ and $\mathbf{G}$ on a particle of mass 1 traveling around the circle $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$ for $0 \leq t \leq 2\pi$.

(c) Find a scalar field $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

Problem 5. Div, Grad, Curl. Consider a scalar field $f(x, y, z)$ and a vector field $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$. Then we define vector fields called the "gradient of f" and the "curl of $\mathbf{F}$":

$$\text{Grad}(f) = \nabla f = \langle f_x, f_y, f_z\rangle,$$
$$\text{Curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle.$$

We also define a scalar field called the "divergence of $\mathbf{F}$":

$$\text{Div}(\mathbf{F}) = \nabla \bullet \mathbf{F} = P_x + Q_y + R_z.$$

(a) Check that $\text{Curl}(\text{Grad}(f)) = \nabla \times (\nabla f) = \langle 0, 0, 0\rangle$.

(b) Check that $\text{Div}(\text{Curl}(\mathbf{F})) = \nabla \bullet (\nabla \times \mathbf{F}) = 0$.

[2] Actually, the choice $f(0, 0, 0) = 0$ is arbitrary. We are just saying that a particle on the ground has zero gravitational potential. Only **changes** in potential energy are physically meaningful.