Math 211 **Summer 2022**
Homework 4 **Drew Armstrong**

Problem 1. Integration over a Rectangle. Let $f(x,y) = 6x^2y$ and consider the rectangle R where $-1 \leq x \leq 1$ and $0 \leq y \leq 4$.

(a) Compute the integral $\iint_R f(x,y)\,dxdy$ by integrating over x first.

(b) Compute the integral $\iint_R f(x,y)\,dxdy$ by integrating over y first. Observe that you get the same answer.

(a): We have

$$\begin{aligned}
\iint_R f\,dA &= \iint_R 6x^2y\,dxdy \\
&= \int_0^4 \left(\int_{-1}^1 6x^2y\,dx \right) dy \\
&= \int_0^4 \left[2x^3y\right]_{-1}^1 \,dy \\
&= \int_0^4 \left[2(1)^2y - 2(-1)^3y\right] \,dy \\
&= \int_0^4 4y\,dy \\
&= \left[2y^2\right]_0^4 \\
&= 32.
\end{aligned}$$

(b): We have

$$\begin{aligned}
\iint_R f\,dA &= \iint_R 6x^2y\,dxdy \\
&= \int_{-1}^1 \left(\int_0^4 6x^2y\,dy \right) xy \\
&= \int_{-1}^1 \left[3x^2y^2\right]_0^4 \,dy \\
&= \int_{-1}^1 \left[3x^2(4)^2 - 3x^2(0)^2\right] \,dy \\
&= \int_{-1}^1 48x^2\,dx \\
&= \left[16x^3\right]_{-1}^1 \\
&= 16(1)^3 - 16(-1)^3 \\
&= 32.
\end{aligned}$$

Remark: Since the integrand $6x^2y$ is separable, we could also write

$$\iint 6x^2y\,dxdy = 6\int_{-1}^1 x^2\,dx \int_0^4 y\,dy$$

$$= 6 \cdot \left[\frac{1}{3}x^3\right]_{-1}^{1} \cdot \left[\frac{1}{2}y^2\right]_{0}^{4} = \cdots = 32.$$

Problem 2. Polar Coordinates. Cartesian coordinates (x, y) and polar coordinates (r, θ) are related as follows:

$$\left\{ \begin{array}{ll} x & = r\cos\theta \\ y & = r\sin\theta \end{array} \right\} \iff \left\{ \begin{array}{ll} r & = \sqrt{x^2+y^2} \\ \theta & = \arctan(y/x) \end{array} \right\}$$

We will use the following notation[1] for the determinants of the Jacobian matrices:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} \quad \text{and} \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \det\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix}.$$

(a) Compute $\partial(x,y)/\partial(r,\theta)$.
(b) Compute $\partial(r,\theta)/\partial(x,y)$ and verify that

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1.$$

(a): First we compute the partial derivatives:

$$\begin{aligned} x_r &= \cos\theta, \\ x_\theta &= -r\sin\theta, \\ y_r &= \sin\theta, \\ y_\theta &= r\cos\theta. \end{aligned}$$

Then we compute the determinant:

$$\begin{aligned} \det\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} &= \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) \\ &= r\cos^2\theta + r\sin^2\theta) \\ &= r\left(\cos^2\theta + \sin^2\theta\right) \\ &= r. \end{aligned}$$

(b): First we compute the partial derivatives:

$$\begin{aligned} r_x &= \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}}, \\ r_y &= \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}}, \\ \theta_x &= \frac{1}{(y/x)^2+1} \cdot \frac{-y}{x^2} = \cdots = \frac{-y}{x^2+y^2}, \\ \theta_y &= \frac{1}{(y/x)^2+1} \cdot \frac{1}{x} = \cdots = \frac{x}{x^2+y^2}. \end{aligned}$$

[1]Warning: Just as dy/dx is not a quotient of numbers, $\partial(x,y)/\partial(r,\theta)$ is not a quotient of numbers. It's just a notation for the determinant of the Jacobian matrix.

Then we compute the determinant:

$$\det\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \det\begin{pmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/(x^2+y^2) & x/(x^2+y^2) \end{pmatrix}$$
$$= \frac{x}{\sqrt{x^2+y^2}}\cdot\frac{x}{x^2+y^2} - \frac{-y}{x^2+y^2}\cdot\frac{y}{\sqrt{x^2+y^2}}$$
$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}$$
$$= \frac{1}{\sqrt{x^2+y^2}}.$$

Since $r = \sqrt{x^2+y^2}$ this implies that

$$\frac{\partial(x,y)}{\partial(r,\theta)}\cdot\frac{\partial(r,\theta)}{\partial(x,y)} = r\cdot\frac{1}{r} = 1,$$

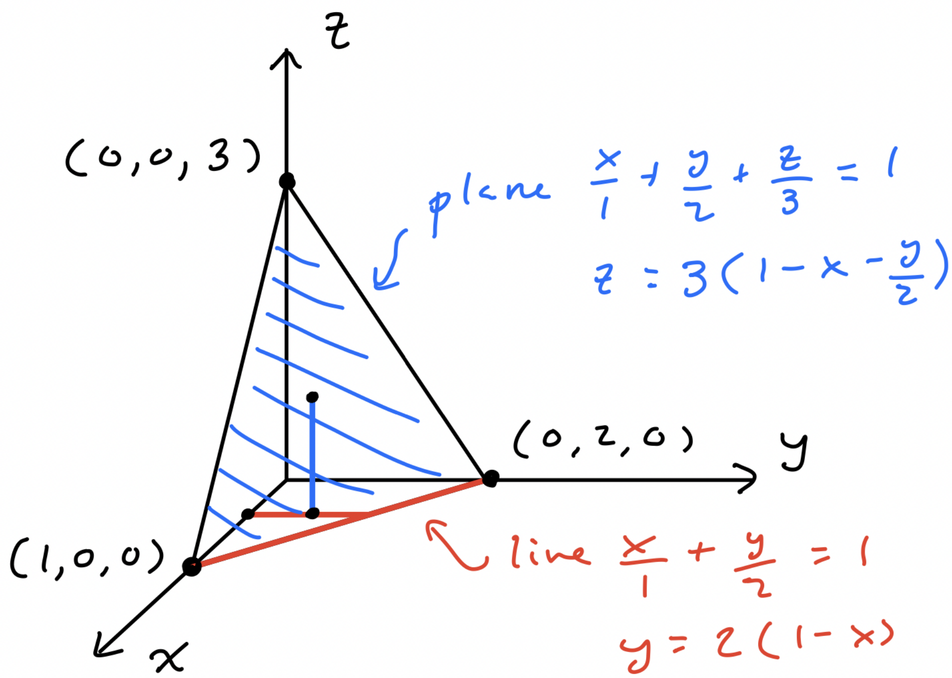
as expected.

Remark: It's pretty cool that we can predict the answer to part (b) without having to do the messy computation.

Problem 3. Integration Over a Tetrahedron. Let E be the solid tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$.

(a) Find a parametrization for this region.
(b) Use your parametrization to compute the volume of E.

(a): First we fix a value of x between 0 and 1. Then y can range between 0 and $2(1-x)$. After choosing y, then z can range between 0 and $3(1-x-y/2)$. Here is a picture:



The red line in the x, y-plane has equation $x/1 + y/2 = 1$ because it has intercepts $(1, 0)$ and $(0, 2)$. The blue plane has equation $x/1 + y/2 + z/3 = 1$ because it has intercepts $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$.

(b): The volume of the tetrahedron is $\iiint_E 1\, dV = \iiint_E 1\, dxdydz$. Because of the parametrization we must integrate over z, then y, then x:

$$
\begin{aligned}
\iiint_E 1\, dxdydz &= \int_0^1 \left(\int_0^{2(1-x)} \left(\int_0^{3(1-x-y/2)} 1\, dz \right) dy \right) dx \\
&= \int_0^1 \left(\int_0^{2(1-x)} 3(1 - x - y/2)\, dy \right) dx \\
&= \int_0^1 \left[3(y - xy - y^3/6) \right]_0^{2(1-x)} dx \\
&= 3 \int_0^1 \left[(1-x)y - y^3/6) \right]_0^{2(1-x)} dx \\
&= 3 \int_0^1 \left[2(1-x)^2 - 8(1-x)^3/6 \right] dx \\
&= 6 \int_0^1 \left[(1-x)^2 - 4(1-x)^3/6 \right] dx \\
&= 6 \cdot \left[\frac{1}{3}(1-x)^3(-1) - \frac{1}{6}(1-x)^4(-1) \right]_0^1 \\
&= 6 \cdot \left[\frac{1}{3} - \frac{1}{6} \right] \\
&= 6 \cdot \frac{1}{6} \\
&= 1.
\end{aligned}
$$

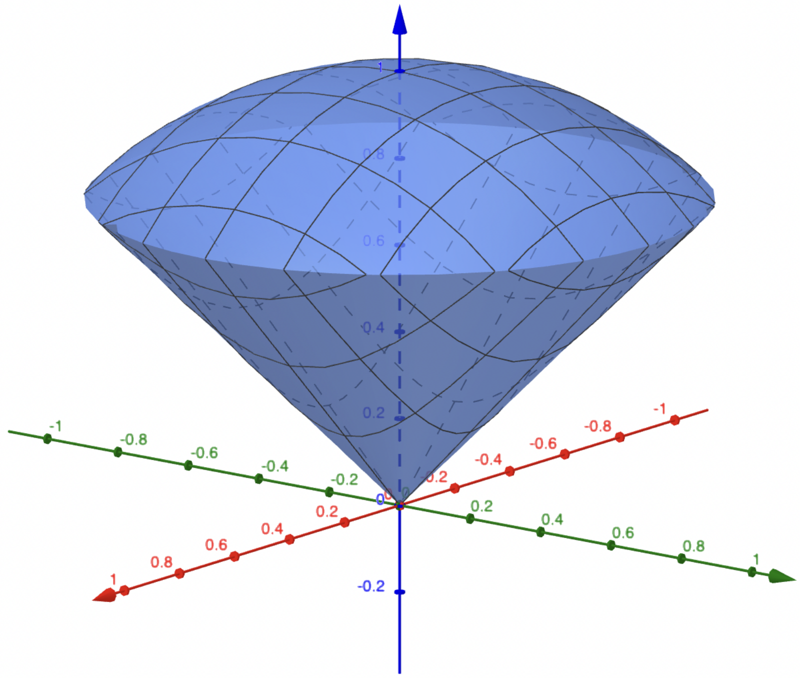
What a nice answer.

Remark: In general, the tetrahedron with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ has volume $abc/6$. The easiest way to prove this is to first prove it for $a = b = c = 1$ and then use a stretching argument as in Problem 5(b).

Problem 4. Spherical Coordinates. Consider the solid region $E \subseteq \mathbb{R}^3$ that is inside the sphere $x^2 + y^2 + z^2 \leq 1$ and above the cone $z^2 = x^2 + y^2$ with $z \geq 0$. Assume that this region has constant density 1 unit of mass per unit of volume.

(a) Use spherical coordinates to compute the mass $m = \iiint_E 1\, dV$.
(b) Compute the moment about the xy-plane, $M_{xy} = \iiint_E z\, dV$, and use this to find the center of mass. [Hint: Because the shape has rotational symmetry around the z-axis we know that $M_{xz} = M_{yz} = 0$.]

(a): Here is a picture of the region:



Note that the cone has slope 1. Indeed, if we set $y = 0$ then the equation of the cone becomes

$$\begin{aligned} z^2 &= x^2 + 0^2 \\ z^2 - x^2 &= 0 \\ (z-x)(z+x) &= 0. \end{aligned}$$

This implies that $z = x$ or $z = -x$, which gives two lines of slope $+1$ and -1 in the xz-plane. This tells us that the angle φ from the vertical goes from 0 to $\pi/4$. The distance ρ from the origin goes from 0 to 1 and the angle θ around the z-axis goes from 0 to 2π. The mass is

$$\begin{aligned} m &= \iiint_E 1\, dV \\ &= \iiint_E \rho^2 \sin\varphi \, d\rho d\theta d\varphi \\ &= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \sin\varphi \, d\varphi \cdot \int_0^1 \rho^2 \, d\rho \\ &= 2\pi \cdot [-\cos\varphi]_0^{\pi/4} \cdot \left[\frac{1}{3}\rho^3\right]_0^1 \\ &= \frac{2}{3}\pi \cdot [-\cos(\pi/4) + \cos(0)] \\ &= \frac{2}{3}\pi \cdot \left[-\frac{\sqrt{2}}{2} + 1\right] \\ &= \frac{\pi}{3}\left(2 - \sqrt{2}\right). \end{aligned}$$

(b): To compute the moment about the xy-plane we use the fact that $z = \rho\cos\varphi$ in spherical coordinates:

$$\begin{aligned}
M_{xy} &= \iiint_E z\,dV \\
&= \iiint_E \rho\cos\varphi\rho^2\sin\varphi\,d\rho d\theta d\varphi \\
&= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \cos\varphi\sin\varphi\,d\varphi \cdot \int_0^1 \rho^3\,d\rho \\
&= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \frac{1}{2}\sin(2\varphi)\,d\varphi \cdot \int_0^1 \rho^3\,d\rho \\
&= 2\pi \cdot \left[-\frac{1}{4}\cos(2\varphi)\right]_0^{\pi/4} \cdot \left[\frac{1}{4}\rho^4\right]_0^1 \\
&= \frac{\pi}{8} \cdot [-\cos(\pi/2) + \cos(0)] \\
&= \frac{\pi}{8} \cdot [-0 + 1] \\
&= \frac{\pi}{8}.
\end{aligned}$$

Hence the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(0, 0, \frac{\pi/8}{\pi(2-\sqrt{2})/3}\right) = (0, 0, 0.64).$$

Problem 5. Volume of an Ellipsoid. Let a, b, c be positive.

(a) Use spherical coordinates to compute the volume of the unit sphere: $x^2 + y^2 + z^2 = 1$.
(b) Use the change of variables $(x, y, z) = (au, bv, cw)$ and part (a) to compute the volume of the ellipsoid: $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.

(a): In spherical coordinates, the unit sphere is described by $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. Hence the volume is[2]

$$\begin{aligned}
\iiint 1\,dV &= \iiint \rho^2\sin\varphi\,d\rho d\theta d\varphi \\
&= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\varphi\,d\varphi \cdot \int_0^1 \rho^2\,d\rho \\
&= 2\pi \cdot [-\cos\varphi]_0^{\pi} \cdot \left[\frac{1}{3}\rho^3\right]_0^1 \\
&= 2\pi \cdot [-(-1) + 1] \cdot \left[\frac{1}{3}\right] \\
&= \frac{4}{3}\pi.
\end{aligned}$$

[2]We could also quote the fact (proved in class) that a sphere of radius R has volume $\frac{4}{3}\pi R^3$.

(b): Consider the change of variables $(x, y, z) = (au, bv, cz)$. The Jacobian determinant is

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \\
&= \det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \\
&= a\det\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} - 0\det\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} + 0\det\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \\
&= abc.
\end{aligned}$$

In other words, the change of variables $(x, y, z) = (au, bv, cz)$ just scales all volumes by abc. This makes sense because we are just scaling the x, y, z-coordinates by a, b, c, respectively.

The volume of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ is

$$\begin{aligned}
\iiint\limits_{(x/a)^2+(y/b)^2+(z/c)^2\leq 1} 1\, dxdydz &= \iiint\limits_{u^2+v^2+w^2\leq 1} abc\, dudvdw \\
&= abc \cdot \iiint\limits_{u^2+v^2+w^2\leq 1} 1\, dudvdw \\
&= abc \cdot \frac{4}{3}\pi.
\end{aligned}$$

Remark: In general, if we scale any solid region by a, b, c in the x, y, z-directions, respectively, then its volume gets multiplied by abc.