Math 211 **Summer 2022**
Homework 3 **Drew Armstrong**

Problem 1. Tangent Line to an Ellipse. Let $a, b > 0$ and consider the ellilpse

$$ax^2 + by^2 = 1.$$

(a) Let $P = (x_0, y_0)$ be a point on the ellipse. Show that the tangent line at P has equation

$$ax_0 x + by_0 y = 1.$$

[Hint: Think of the ellipse as the level curve $f(x, y) = 1$ where $f(x, y) = ax^2 + by^2$.]

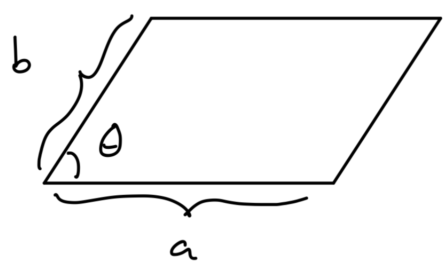
(b) Draw a picture of the ellipse and tangent line when $a = 1$, $b = 3$ and $P = (1/2, 1/2)$.

Problem 2. Multivariable Chain Rule Practice. Let $f(x, y)$ be a function of x and y, where $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$ are functions of r and θ.

(a) Express f_r and f_θ in terms of r, θ, f_x and f_y.

(b) Express f_{rr} in terms of $r, \theta, f_{xx}, f_{yy}$ and f_{xy}. [Hint: Use the formulas $f_{xr} = f_{xx}\frac{dx}{dr} + f_{xy}\frac{dy}{dr} = f_{xx}\cos\theta + f_{xy}\sin\theta$ and $f_{yr} = f_{yx}\frac{dx}{dr} + f_{yy}\frac{dy}{dr} = f_{yx}\cos\theta + f_{yy}\sin\theta$.]

Problem 3. Linear Approximation. Consider a parallelogram with side lengths a, b and angle θ as follows:



(a) Find a formula for the area $A(a, b, \theta)$ in terms of a, b and θ.

(b) Suppose that we measure a, b, θ with the following uncertainties:

$$\begin{aligned} a &= 2 \pm 0.1 \text{ cm}, \\ b &= 1 \pm 0.1 \text{ cm}, \\ \theta &= 45 \pm 1 \text{ degrees}. \end{aligned}$$

Use these measurements together with your formula from part (a) to estimate the area.

Problem 4. Constrained Optimization. Let $f(x, y) = xy$ be a temperature distribution in the plane. Suppose that you travel around the unit circle $x^2 + y^2 = 1$ with parametrization $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ from $t = 0$ to $t = 2\pi$.

(a) Let $T(t) = f(\mathbf{r}(t))$ be your temperature at time t. Compute $T'(t)$.

(b) Find all times t where $T(t)$ is maximized or minimized. [Hint: Set $T'(t) = 0$.]

(c) Use part (b) to find all points on the unit circle where the temperature is maximized or minimized.

(d) *Method of Lagrange Multipliers.* Now we express the circle as $g(x, y) = 1$ where $g(x, y) = x^2 + y^2$. Find all points on the circle where the vectors $\nabla g(x, y)$ and $\nabla f(x, y)$ point in the same direction. [Hint: Let $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some scalar λ. Use this and the equation $x^2 + y^2 = 1$ to solve for x and y. It's not as hard as it looks.]

Problem 5. Unconstrained Optimization. Let $f(x, y) = x^3 + 2xy - 4y^2 - 6x$ be a temperature distribution in the plane.

(a) Compute the gradient vector $\nabla f(x, y)$ and the Hessian determinant $\det(Hf)$.

(b) Find all critical points (x, y), i.e., all points where the gradient vector is zero:

$$\nabla f(x, y) = \langle 0, 0 \rangle.$$

(c) Use the "second derivative test" to determine whether each critical point is a local maximum, local minimum, saddle point, or none of the above.