HW 2 due Friday 11:40am.

Quiz 2 Tuesday 11:40am.

Recall: Suppose a particle in $\mathbb{R}^2$ has constant acceleration vector

$$\vec{a}(t) = \langle a, b \rangle \text{ for all } t.$$

If the initial position & velocity are

$$\vec{r}(0) = \langle x_0, y_0 \rangle$$
$$\vec{v}(0) = \langle u_0, v_0 \rangle$$

Then we can "integrate the system":

$$\vec{v}(t) = \int \vec{a}(t)\,dt$$

$$= \langle at + c_1, bt + c_2 \rangle$$

When $t = 0$ we find that

$$\langle u_0, v_0 \rangle = \vec{v}(0) = \langle 0 + c_1, 0 + c_2 \rangle,$$

so $\vec{v}(t) = \langle at + u_0, bt + v_0 \rangle$.

Then we have

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= \left\langle \frac{1}{2}at^2 + u_0 t + c_3 \,,\, \frac{1}{2}bt^2 + v_0 t + c_4 \right\rangle$$

When $t=0$ we find that

$$\langle x_0, y_0 \rangle = \vec{r}(0) = \langle 0 + c_3 \,,\, 0 + c_4 \rangle$$

we conclude that

$$\vec{r}(t) = \left\langle \frac{1}{2}at^2 + u_0 t + x_0 \,,\, \frac{1}{2}bt^2 + v_0 t + y_0 \right\rangle$$

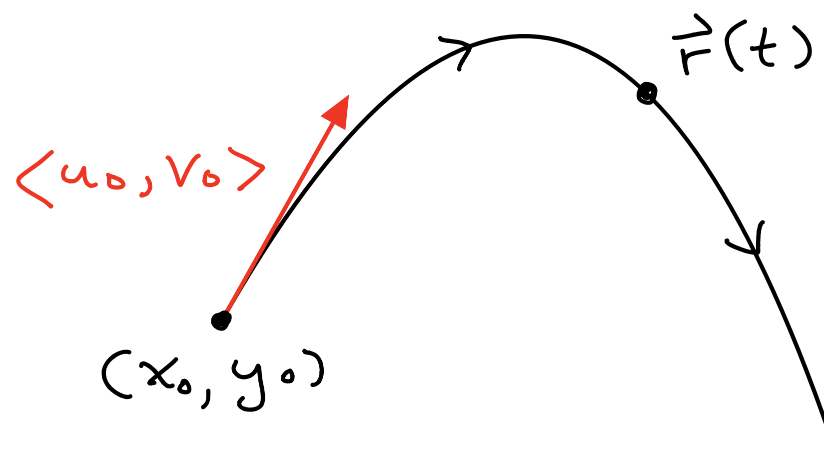
Example: The acceleration due to gravity near the earth's surface is approximately constant:

$$\vec{a}(t) \approx \langle 0, -32 \text{ ft/sec}^2 \rangle.$$

So the above formula gives

$$\vec{r}(t) = \langle u_0 t + x_0 \,,\, -16t^2 + v_0 t + y_0 \rangle.$$

This is a parametrized parabola:



Today we will discuss a more interesting example where the acceleration vector is

not constant.

Universal Gravitation:

We have two points with mass M & m, say the sun has mass M and a planet has mass m. Fix the sun at the origin $\langle 0,0,0 \rangle$

and let the planet have position

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

Newton says that the planet experiences a force due to gravity:

- $\| \vec{F}(t) \| = GMm / \| \vec{r}(t) \|^2$
- $\vec{F}(t)$ in the direction of $-\vec{r}(t)$.

If we let $\vec{u}(t) = \vec{r}(t) / \| \vec{r}(t) \|$ be a unit vector in the direction of $\vec{r}(t)$ [see HW2.5] then we have $\| \vec{u}(t) \| = 1$ for all t, and

$$\vec{F}(t) = -\frac{GMm}{\| \vec{r}(t) \|^2} \vec{u}(t)$$

$$= -\frac{GMm}{\| \vec{r}(t) \|^3} \vec{r}(t)$$

From Newton's Second Law, this causes the planet to accelerate:

$$m\vec{a}(t) = \vec{F}(t)$$

$$\vec{a}(t) = \frac{1}{m}\vec{F}(t)$$

$$\vec{a}(t) = -\frac{GM}{\|\vec{r}(t)\|^2}\vec{u}(t)$$

$$\text{or} \quad -\frac{GM}{\|\vec{r}(t)\|^3}\vec{r}(t)$$

To simplify notation we will assume that $GM = 1$, we will write $\|\vec{r}(t)\| = r(t)$ and we will omit the "t" whenever possible:

$$\vec{a} = -\frac{1}{r^2}\vec{u} = -\frac{1}{r^3}\vec{r}$$

This is called a "differential eqn."

$$\vec{r}'' = -\frac{1}{r^3}\vec{r}.$$

We want to solve for

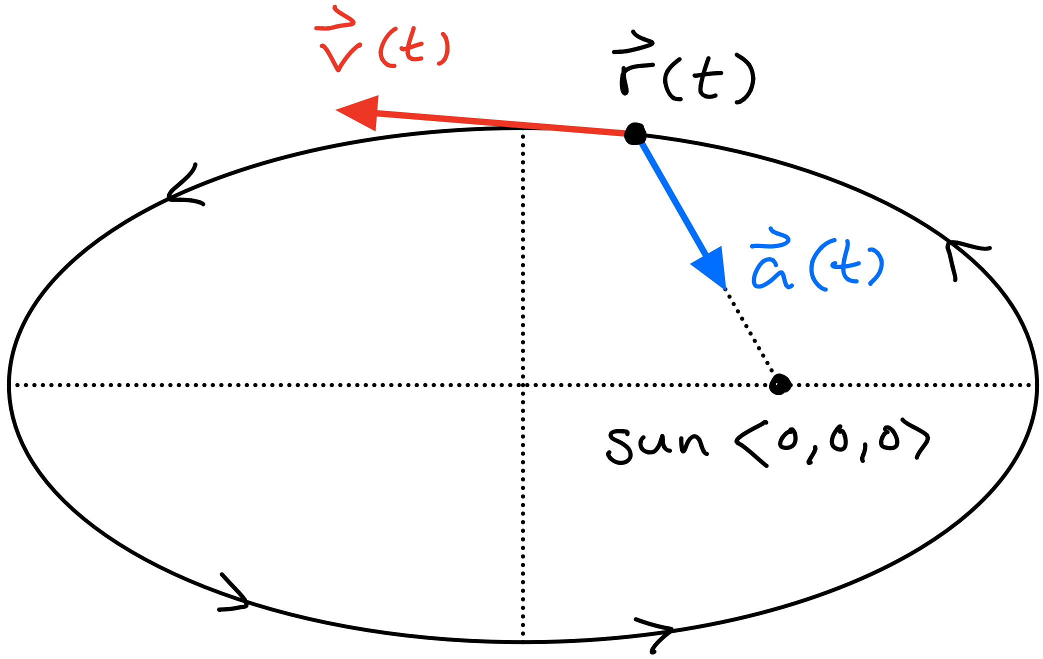
$$\vec{r} = \vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

[Remark : $r(t) = \|\vec{r}(t)\|$
$= \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$.]

That is, we want to compute the orbit of the planet.

Kepler observed experimentally that planetary orbits are <u>ellipses</u> with sun at one "focus". Newton used his "inverse square law" to <u>prove this</u> !

We will find that the picture looks like this :



The proof is "a bit involved" but I want to show you because it is is a beautiful application of vector calculus. We will need to use the following rules:

- $\vec{x} \times \vec{x} = \vec{0}$
- if $\|\vec{x}(t)\| =$ constant, then $\vec{x}(t) \cdot \vec{x}\,'(t) = 0$ for all times t.

[See HW 2.5]

We will also need some properties of the cross product

- $\vec{x} \cdot (\vec{y} \times \vec{z}) = (\vec{x} \times \vec{y}) \cdot \vec{z}$
- $\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{x} \cdot \vec{y})\vec{z}$,

and the "product rules"

- $[f(t)\vec{x}(t)]' = f'(t)\vec{x}(t) + f(t)\vec{x}'(t)$
- $[\vec{x}(t) \cdot \vec{y}(t)]' = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot y'(t)$
- $[\vec{x}(t) \times \vec{y}(t)]' = \vec{x}'(t) \times \vec{y}(t) + \vec{x}(t) \times \vec{y}'(t)$.

Let's begin. We have defined

$$\vec{u} = \frac{1}{r}\vec{r}$$

and we know the inverse square law

$$\vec{a} = \vec{r}'' = -\frac{1}{r^2}\vec{u} = -\frac{1}{r^3}\vec{r}$$

First we observe that

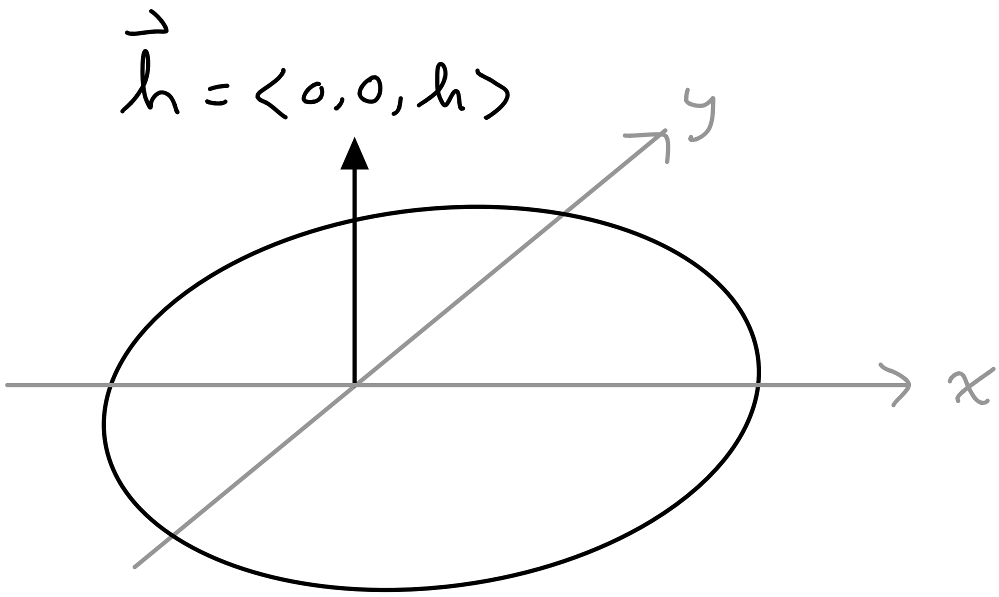
$$
\begin{aligned}
(\vec{r} \times \vec{v})' &= \vec{r}\,' \times \vec{v} + \vec{r} \times \vec{v}\,' \\
&= \vec{v} \times \vec{v} + \vec{r} \times \vec{a} \\
&= \cancel{\vec{v} \times \vec{v}} - \frac{1}{r^3}\, \cancel{\vec{r} \times \vec{r}} \\
&= \vec{0} - \vec{0} \\
&= \vec{0}
\end{aligned}
$$

This tells us that the vector $\vec{r}(t) \times \vec{v}(t)$ is constant, i.e., does not depend on t. Let's choose a coordinate system so

$$\vec{r}(t) \times \vec{v}(t) = \vec{h} = \langle 0, 0, h \rangle$$

for some constant $h > 0$.

Since $\vec{r} \times \vec{v}$ is simultaneously perpendicular to $\vec{r}$ & $\vec{v}$, it follows that the planet stays in the x,y-plane:



Next, I claim that

$$(\vec{v} \times \vec{h})' = \vec{u}' = \left(\frac{\vec{r}}{r}\right)'$$

This is the tricky part! To prove it we will express everything in terms of $\vec{u}(t)$:

$$\vec{a} = -\frac{1}{r^2}\vec{u}$$

$$\vec{r} = r\vec{u}$$

$$\vec{v} = (r\vec{u})'$$
$$= r\vec{u}\,' + r'\vec{u}$$

and

$$\vec{h} = \vec{r} \times \vec{v}$$
$$= (r\vec{u}) \times (r\vec{u}\,' + r'\vec{u})$$
$$= r^2(\vec{u} \times \vec{u}\,') + rr'\underbrace{\cancel{(\vec{u} \times \vec{u})}}_{\vec{0}}$$
$$= r^2(\vec{u} \times \vec{u}\,').$$

Thus we can prove the desired identity:

$$(\vec{v} \times \vec{h})' = \vec{v}\,' \times \vec{h} + \vec{v} \times \underbrace{\cancel{\vec{h}\,'}}_{\vec{0}}$$

$\vec{h}$ is constant

$$= \vec{v}\,' \times \vec{h} + \vec{0}$$
$$= \vec{a} \times \vec{h}$$
$$= \left(-\frac{1}{\cancel{r^2}}\vec{u}\right) \times \left(\cancel{r^2}(\vec{u} \times \vec{u}\,')\right)$$

$$= -\vec{u} \times (\vec{u} \times \vec{u}')$$

$$= (\cancel{\vec{u} \cdot \vec{u}}^{\,1})\,\vec{u}' - (\cancel{\vec{u} \cdot \vec{u}'}^{\,0})\,\vec{u}$$

[definition] [HW 2.5]

$$= \vec{u}' \text{ as desired } \checkmark$$

That was the hardes part.
The rest is downhill.

By integrating

$$(\vec{v} \times \vec{h})' = \vec{u}'$$

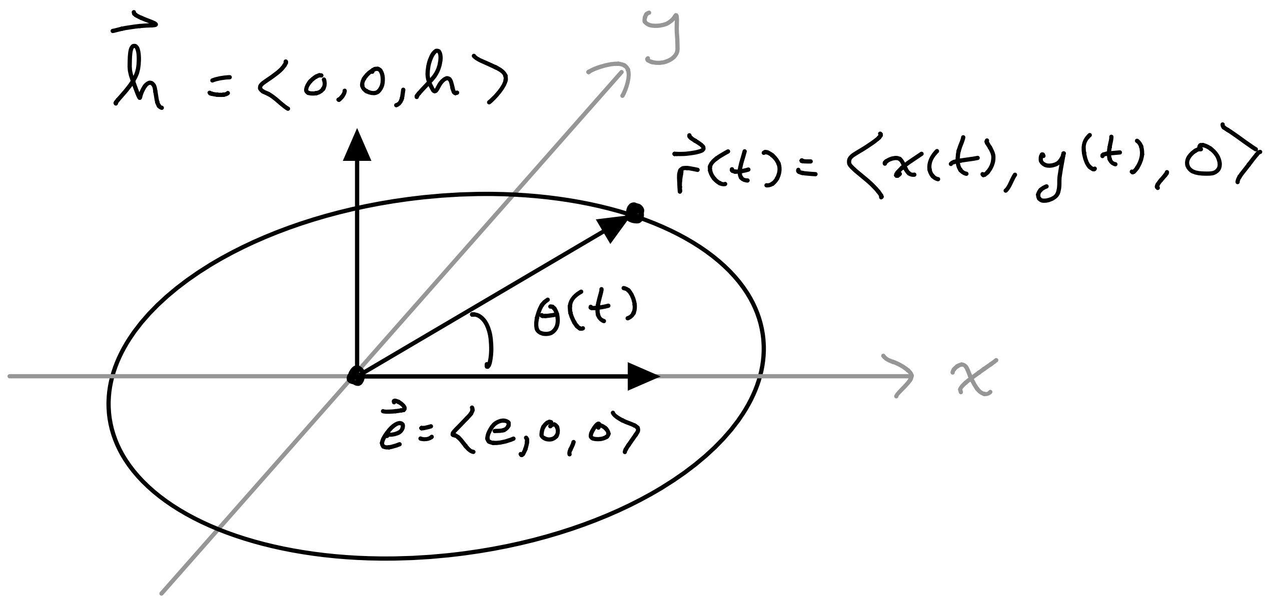
we obtain

$$\vec{v} \times \vec{h} = \vec{u} + \vec{e}$$

for some constant vector $\vec{e}$.
By choosing coordinates we can assume that

$$\vec{e} = \langle e, 0, 0 \rangle \text{ for some } e \geq 0.$$

Picture :



The position of the planet has the form

$$\vec{r}(t) = \langle x(t), y(t), 0 \rangle .$$

If we let $\theta(t)$ be the angle between $\vec{r}(t)$ & $\vec{e}$ then we can write

$$\begin{cases} x(t) = r(t) \cos \theta(t) \\ y(t) = r(t) \sin \theta(t) \end{cases}$$

in "polar coordinates". To find an equation for x & y we first find an equation for r & θ.

To do this we compute the number $\vec{r}\cdot(\vec{v}\times\vec{h})$ in two different ways:

1) $$\vec{r}\cdot(\vec{v}\times\vec{h}) = \vec{r}\cdot(\vec{u}+\vec{e})$$
$$= \vec{r}\cdot\vec{u} + \vec{r}\cdot\vec{e}$$
$$= (r\vec{u})\cdot\vec{u} + \vec{r}\cdot\vec{e}$$

dot product identity

$$= r(\cancelto{1}{\vec{u}\cdot\vec{u}}) + re\cos\theta$$
$$= r + re\cos\theta$$
$$= r(1+e\cos\theta)$$

2) $$\vec{r}\cdot(\vec{v}\times\vec{h}) = (\vec{r}\times\vec{v})\cdot\vec{h}$$
$$= \vec{h}\cdot\vec{h}$$
$$= \|h\|^2 = h^2.$$

Combining ① and ② gives

$$r(1+e\cos\theta) = h^2$$

$$\boxed{r = \frac{h^2}{1+e\cos\theta}}$$

This is the desired equation for the shape of the orbit.

Believe it or not, this is the equation of a "conic section" in polar coordinates. To see this we will convert to the more familiar cartesian coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

• If $e = 0$ then this is easy. We have

$$r = h^2$$
$$r^2 = h^4$$
$$x^2 + y^2 = h^4,$$

which is a circle of radius h^2.

To clean it up, let $c = h^2$ so we get a circle of radius c:

$$x^2 + y^2 = c^2.$$

• If $0 < e < 1$ then with "a bit of effort" one can verify that

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where the constants a, b, c, d are related to the constants h, e by

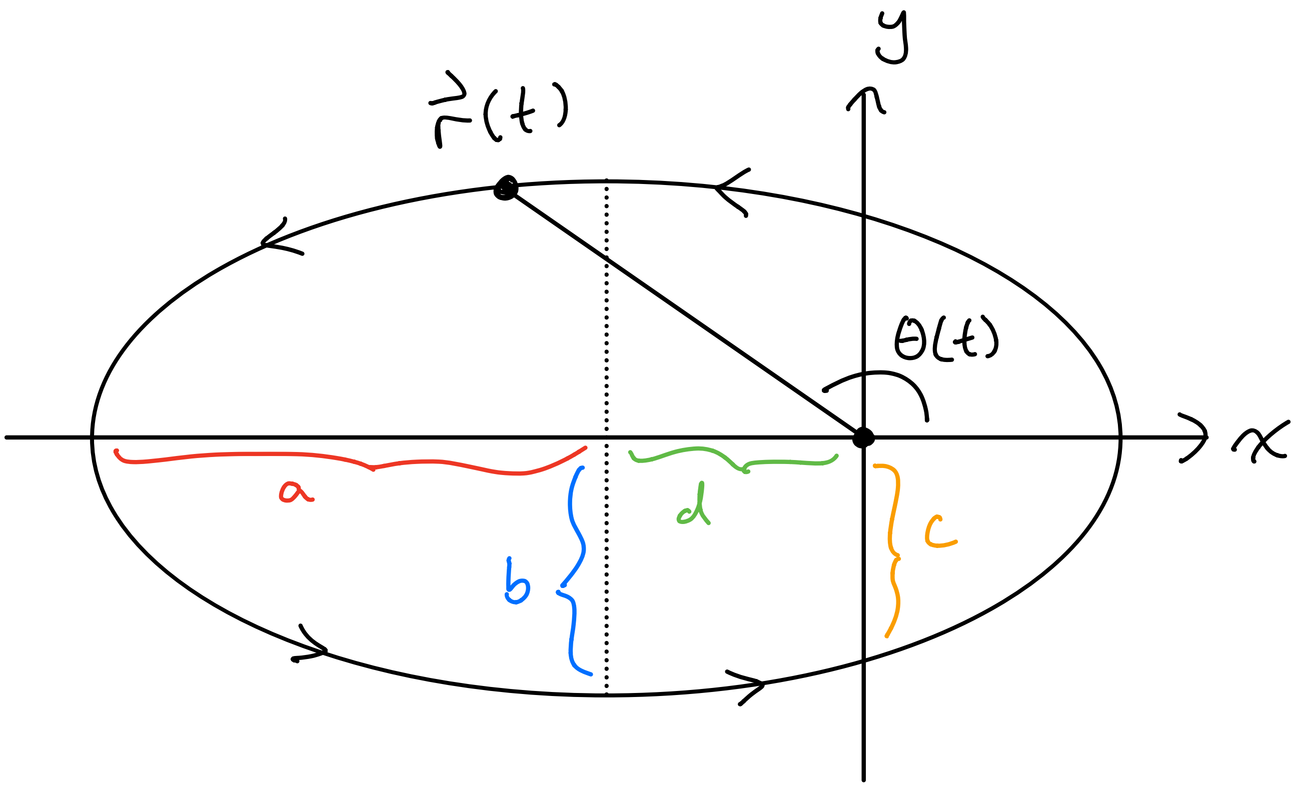
$$c = h^2$$

$$a = c/(1-e^2)$$

$$b = c/\sqrt{1-e^2}$$

$$d = ae$$

This is the familiar equation of an ellipse:



Remark: Sadly, there are no nice formulas for the coordinates $r(t)$, $\theta(t)$, $x(t)$, $y(t)$ as functions of t. I presume that the formulas would involve "elliptic functions" just as the arc length of an ellipse does.

- If $e \geq 1$ then the planet has enough energy to escape the solar system: