HW 2 is posted; due Fri 11:40am.

Quiz 2: Tues, June 1st

(due to no class on Mon May 31)

Review: A function $\vec{r}: \mathbb{R} \to \mathbb{R}^n$ sends a single input t to a vector of outputs $\vec{r}(t)$, which we can write as

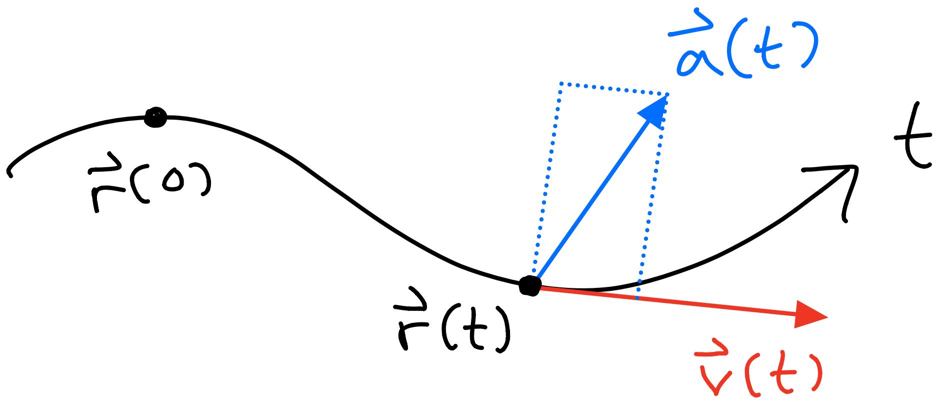
$$\vec{r}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle$$

We can view this as a "parametrized curve" with velocity & acceleration vectors:

$$\vec{v}(t) = \vec{r}\,'(t) = \langle x_1'(t), \ldots, x_n'(t) \rangle$$

$$\vec{a}(t) = \vec{v}\,'(t) = \langle x_1''(t), \ldots, x_n''(t) \rangle$$

The velocity is tangent to the curve. The acceleration vector tells us how the curve is "turning":



Strictly speaking: The acceleration has a component in the tangent direction that changes the <u>speed</u> & a component in a normal direction that causes <u>curving</u>.

Newton's Second Law:

A force $\vec{F}$ acting on a particle of mass m causes an acceleration $\vec{a}$ according to:

$$\vec{F} = m\vec{a}$$

Example : Projectile Motion.

Galileo says that the acceleration of a particle due to gravity is constant & independent of mass.

In a vertical plane we have

$$\vec{a} \approx \langle 0, -32 \text{ feet/sec}^2 \rangle$$

If a projectile is launched from $\langle 0,0\rangle$ with initial velocity $\langle 20, 80\rangle$, find the trajectory

$$\vec{r}(t) = \langle x(t), y(t)\rangle .$$

- When does it hit the ground?
- Where does it land?
- When does it reach max. height?
- How high does it go?

Solution : We are given

$$\vec{r}(0) = \langle 0, 0 \rangle$$

$$\vec{v}(0) = \langle 20 \text{ ft/s}, 80 \text{ ft/s} \rangle$$

$$\vec{a}(t) = \langle 0, -32 \text{ ft/s}^2 \rangle$$

[for all times t]

First we obtain $\vec{v}(t)$ by integrating $\vec{a}(t) = \vec{v}\,'(t)$:

$$\vec{v}(t) = \int \vec{a}(t)\, dt$$

$$= \left\langle \int 0\, dt, \int -32\, dt \right\rangle$$

$$= \langle c_1, -32t + c_2 \rangle$$

for some constants c_1 & c_2.

Plug in the initial velocity:

$$\langle 20, 80 \rangle = \vec{v}(0)$$

$$= \langle c_1, -32 \cdot 0 + c_2 \rangle$$

$$\rightsquigarrow \quad c_1 = 20,$$

$$c_2 = 80.$$

Thus the particle has velocity

$$\vec{v}(t) = \langle 20, -32t + 80 \rangle$$

at time t. [Note that the horiz. speed never changes. This is because the acceleration has no horizontal component.]

To find the position $\vec{r}(t)$ we integrate the velocity $\vec{v}(t) = \vec{r}\,'(t)$:

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= \left\langle \int 20\,dt, \int (-32t + 80)\,dt \right\rangle$$

$$= \langle 20t + c_3, -16t^2 + 80t + c_4 \rangle$$

Plug in the initial position:

$$\langle 0, 0 \rangle = \vec{r}(0)$$

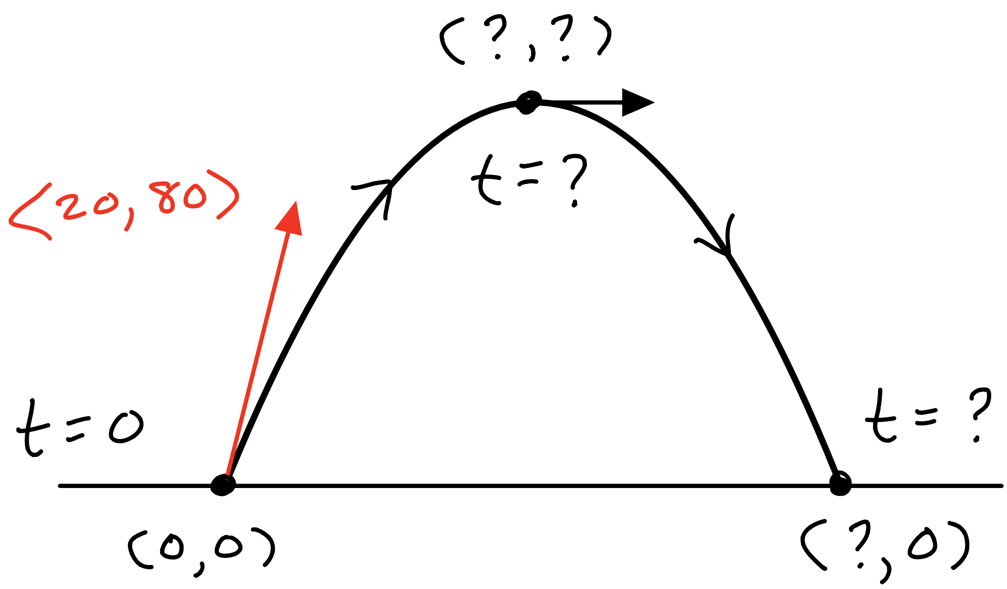
$$= \langle 20 \cdot 0 + c_3, -16 \cdot 0^2 + 8 \cdot 0 + c_4 \rangle$$

$$\longrightarrow \quad c_3 = 0,$$
$$c_4 = 0.$$

Finally, we obtain

$$\vec{r}(t) = \langle 20t, -16t^2 + 80t \rangle$$

This is a parametrized parabola:



- When does it hit the ground?

$$\text{height} = 0$$
$$y(t) = 0$$
$$-16t^2 + 80t = 0$$

$$t(-16t + 80) = 0$$

$t = 0$ or $-16t + 80 = 0$

not interesting

$$t = 80/16$$

$$t = 5 \text{ seconds}$$

- Where does it land?

The position when it hits the ground is

$$\vec{r}(5) = \langle 20 \cdot 5, 0 \rangle$$

$$= \langle 100 \text{ ft}, 0 \rangle$$

- When does it reach max height?

The velocity $\vec{v}(t) = \langle 20, -32t + 80 \rangle$ is horizontal when

$$-32t + 80 = 0$$

$$t = 80/32 = 2.5 \text{ seconds}$$

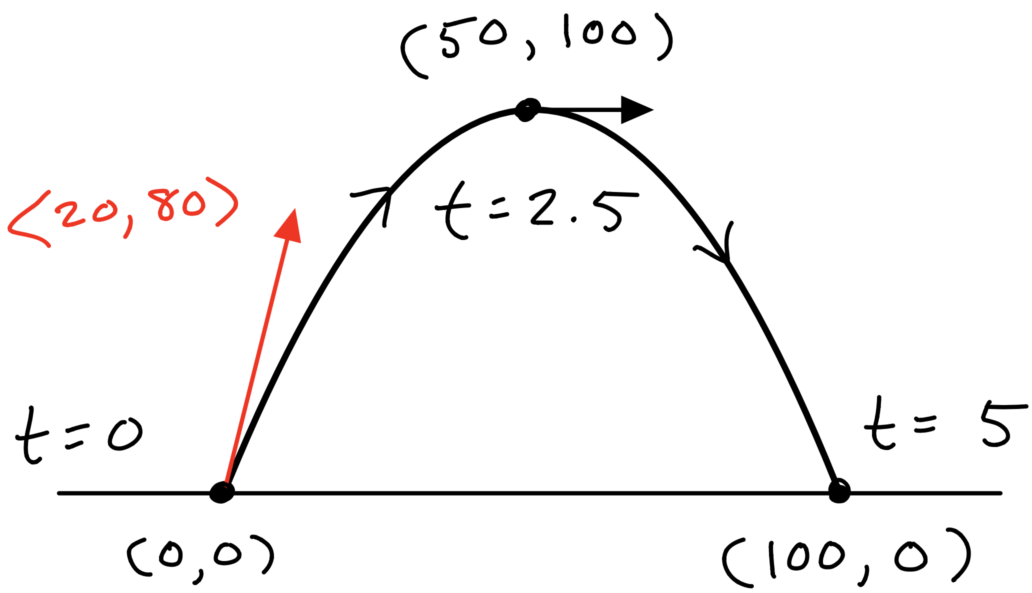
• How high does it go?

The position at time $t = 2.5$ is

$$\vec{r}(2.5) = \langle 20(2.5), -16(2.5)^2 + 80(2.5) \rangle$$

$$= \langle 50, 100 \rangle$$

So the max. height is 100 ft.



[Remark: You may have seen a similar picture in Calc I, but then the motion was only vertical and the horizontal axis represented <u>time</u>, not <u>horiz. dist.</u>]

More generally, if we have

$\vec{a}(t) = \langle 0, -g \rangle$ const. acceleration

$\vec{v}(0) = \langle u_0, v_0 \rangle$ initial velocity

$\vec{r}(0) = \langle x_0, y_0 \rangle$, initial position

then following the same procedure:

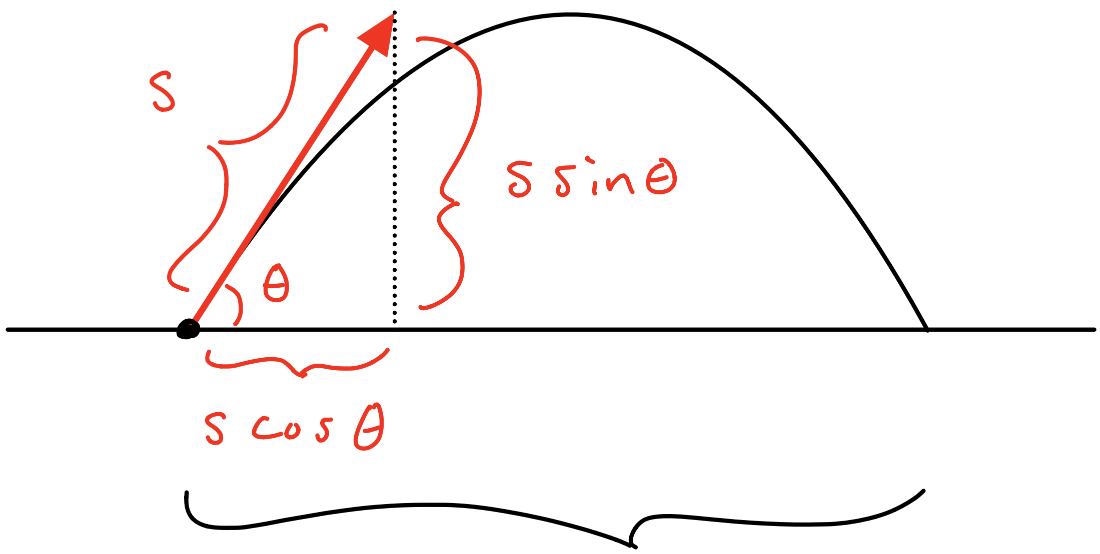
$$\vec{v}(t) = \langle u_0, -gt + v_0 \rangle$$

$$\vec{r}(t) = \langle u_0 t + x_0, -\tfrac{1}{2}gt^2 + v_0 t + y_0 \rangle$$

It turns out that projectile motion in 3D always occurs in a plane, so we don't really need a 3rd coord.

[On HW 2 you use polar coords. for the initial velocity

$$\vec{v}(0) = \langle s\cos\theta, s\sin\theta \rangle.$$



this distance is a function of s & θ.

For given speed s you want to find the angle θ that maximizes the horizontal distance traveled.]

That's how gravity works near the surface of a planet. Out in space it works differently.

Newton's Law of Universal Gravitation:

Given two points of mass M, m separated by distance r, they exert equal and opposite forces on each other of magnitude

$$F = \frac{GMm}{r^2}$$

"The inverse square law"

where G is just some universal constant of nature.

In terms of vectors: Place the sun at the origin $\langle 0,0,0 \rangle$ and the earth at point

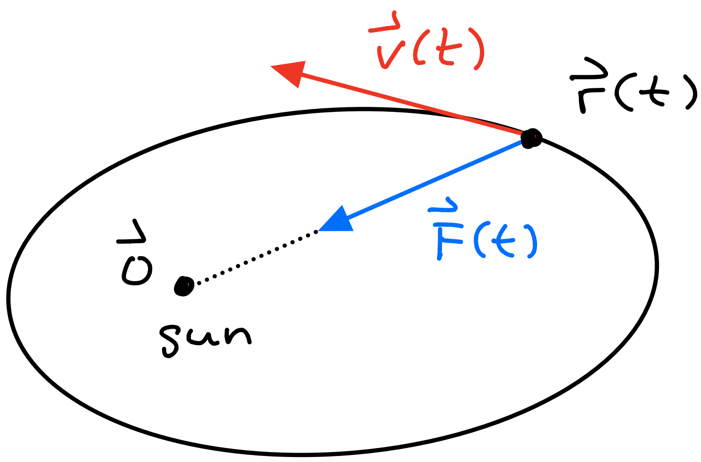
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

If M = mass of sun
m = mass of earth,

the earth feels a force $\vec{F}(t)$ of magnitude

$$\|\vec{F}(t)\| = \frac{GMm}{\|\vec{r}(t)\|^2}$$

pointed in the direction $-\vec{r}$,
i.e., directly toward the sun.



$\vec{F}(t) = m\vec{a}(t)$

Problem: Given the initial position $\vec{r}(0)$ & velocity $\vec{v}(0)$, compute the orbit $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Newton solved this problem in spectacular fashion & he proved that the orbit is an ellipse with the sun at one "focus".

We'll solve this next time. The solution involves the calculus of parametrized curves (pg 272):

- $[c\vec{r}(t)]' = c\vec{r}\,'(t)$
- $[\vec{r}(t) \pm \vec{u}(t)]' = \vec{r}\,'(t) \pm \vec{u}\,'(t)$
- $[f(t)\vec{r}(t)]' = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$
- $[\vec{r}(f(t))]' = \vec{r}\,'(f(t))\, f'(t)$
- $[\vec{r}(t) \bullet \vec{u}(t)]' = \vec{r}\,'(t) \bullet \vec{u}(t) + \vec{r}(t) \bullet \vec{u}\,'(t)$

These last 3 are generalizations of product & chain rules from Calc I. If we are working in 3D, then we also have

- $[\vec{r}(t) \times \vec{u}(t)]' = \vec{r}\,'(t) \times \vec{u}(t) + \vec{r}(t) \times \vec{u}\,'(t)$