Quiz 1 will be graded & HW 2 will be posted tomorrow.

Current Topic: Vector-valued functions & motion in space.

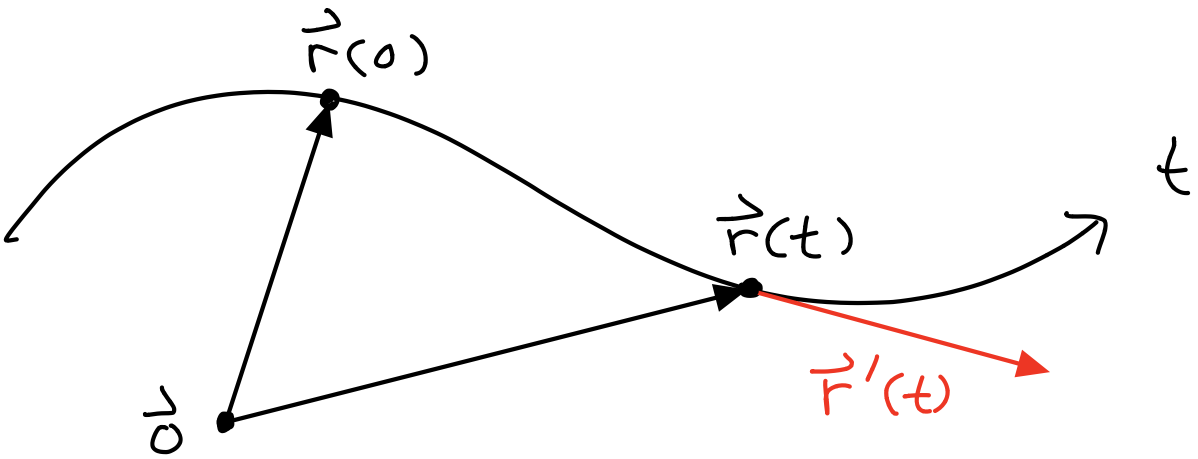
A vector valued function $\mathbb{R} \to \mathbb{R}^n$ sends each number $t \in \mathbb{R}$ to some vector $\vec{r}(t) \in \mathbb{R}^n$ in n-dim. space.

[The symbol "$\in$" means "is a member of the set", so "$t \in \mathbb{R}$" means "t is a real number".]

We can think of the ith component of the vector $\vec{r}(t)$ as a usual ("scalar valued") function of t. Let's call it $x_i(t)$. Thus we have the following notation for a function $\vec{r}$ from $\mathbb{R} \to \mathbb{R}^n$:

$$\boxed{\vec{r}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle}$$

If we want, we can view the vector $\vec{r}(t)$ as the position of a particle at time t:



As before, we define the derivative of a vector-valued function as

$$\vec{r}\,'(t) = \langle x_1'(t), x_2'(t), \ldots, x_n'(t) \rangle$$

$$\frac{d}{dt}\vec{r}(t) = \left\langle \frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt} \right\rangle$$

[The derivative of a function $\vec{r}: \mathbb{R} \to \mathbb{R}^n$ is another function

$\vec{r}\,' : \mathbb{R} \to \mathbb{R}^n$ of the same kind.]

We define the speed at time t,

$$\|\vec{r}\,'(t)\| = \sqrt{(x_1'(t))^2 + \cdots + (x_n'(t))^2},$$

and the arc length between times $t = a$ & $t = b$:

$$\text{arc length} = \int \text{speed}\, dt$$

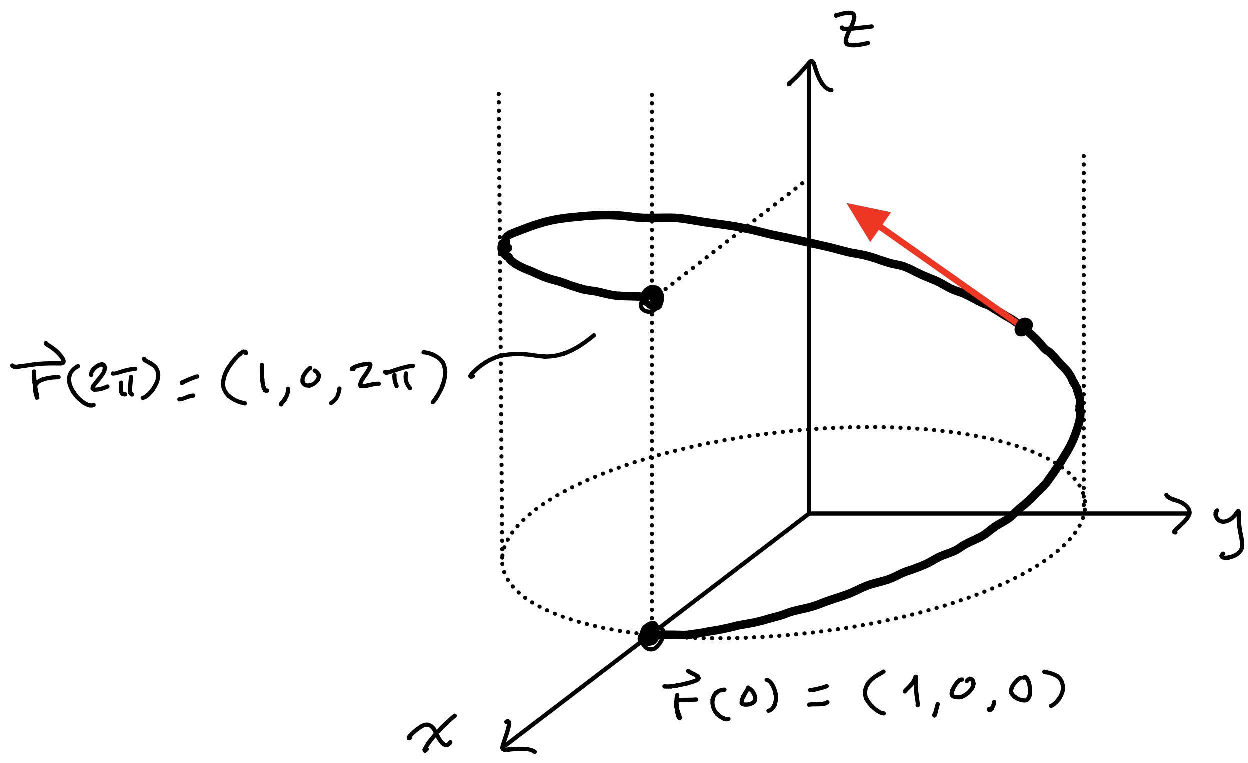
$$= \int_a^b \|\vec{r}\,'(t)\|\, dt.$$

We did this before in $\mathbb{R}^2$. Now we can do it in any number of dimensions.

Example in $\mathbb{R}^3$: Consider the following "helical" path

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$= \langle \cos t, \sin t, t \rangle$$



Find the arc length between times $t = 0$ & $t = 2\pi$.

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{2}$$

The speed is constant.

So the arc length is

$$\int_0^{2\pi} \text{speed}\, dt = \int_0^{2\pi} \sqrt{2}\, dt$$

$$= 2\pi\sqrt{2}$$

$$= \sqrt{2}\,(\text{circumference of unit circle})$$

We can also <u>integrate</u> vector-valued functions. As with derivatives, we do this "component-by-component":

$$\vec{r}(t) = \langle x_1(t), \ldots, x_n(t) \rangle$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b x_1(t)\,dt, \ldots, \int_a^b x_n(t)\,dt \right\rangle$$

[Observe that $\int \vec{r}(t)\,dt$ is also a vector!]

But why do we want to do this?

Newton's 2nd Law

If $\vec{r}(t)$ is the position of a particle at time t then we define the velocity vector & the acceleration vector at time t:

$$\vec{v}(t) = \vec{r}\,'(t),$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t).$$

Examples:

- The parametrized line in $\mathbb{R}^3$

$$\vec{r}(t) = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

has velocity vector

$$\vec{v}(t) = \vec{r}\,'(t)$$

$$= \left\langle \frac{d}{dt}(x_0 + ta), \frac{d}{dt}(y_0 + tb), \frac{d}{dt}(z_0 + tc) \right\rangle$$

$$= \langle a, b, c \rangle,$$

which is <u>constant</u>.

[e.g. $\vec{r}(t) = \langle 2+t, 3-2t, 4+3t \rangle$

$\vec{r}\,'(t) = \langle 1, -2, 3 \rangle.$]

The acceleration vector is

$$\vec{a}(t) = \vec{v}\,'(t)$$

$$= \frac{d}{dt} \langle a, b, c \rangle$$

$$= \left\langle \frac{d}{dt} a, \frac{d}{dt} b, \frac{d}{dt} c \right\rangle$$

$$= \langle 0, 0, 0 \rangle,$$

i.e. the particle is <u>not</u> accelerating.

- The parametrized circle

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

has velocity vector

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

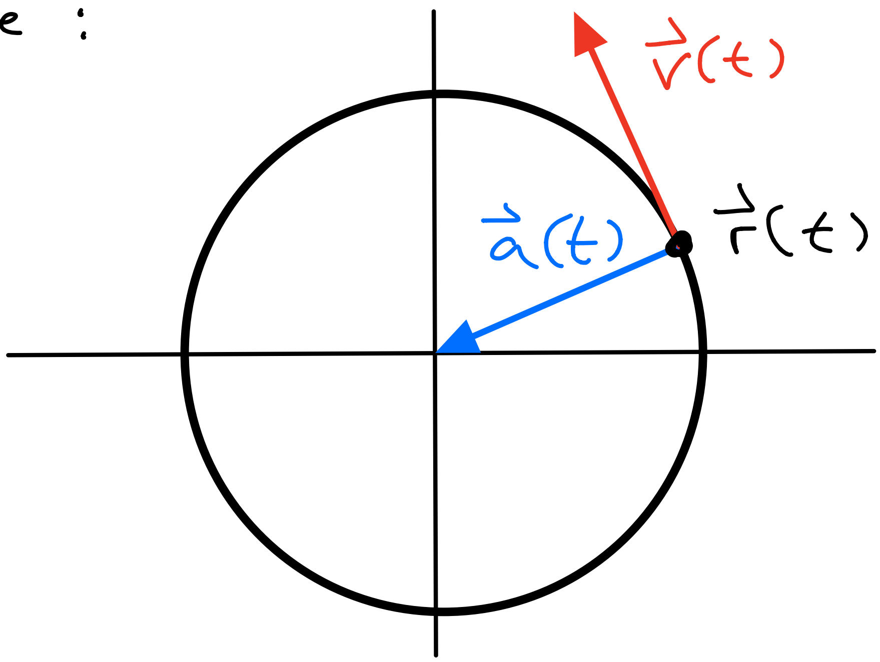
[speed $\|\vec{v}(t)\| = 1$ is constant]

and acceleration vector

$$\vec{a}(t) = \vec{v}\,'(t) = \langle -\cos t, -\sin t \rangle$$

[note that $\vec{a}(t) = -\vec{r}(t)$]

Picture :



The velocity is tangent to the circle. The acceleration always points toward the origin!

[The acceleration doesn't change the speed (the length of the velocity), it changes the direction of the velocity vector.]

- The helical path

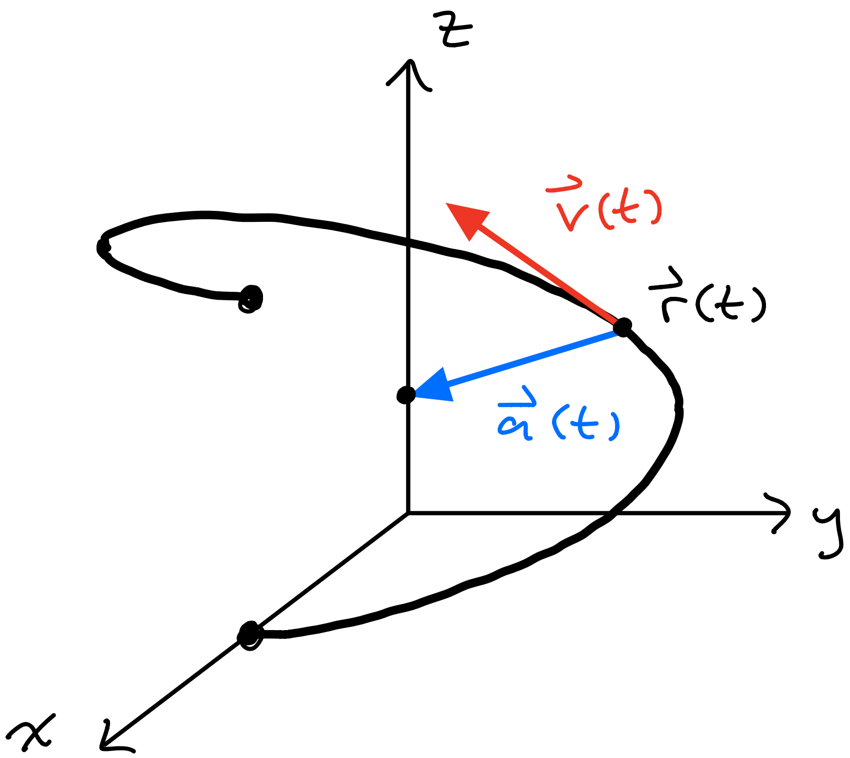
$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

has velocity & acceleration

$$\vec{v}(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{a}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

Picture:



The acceleration always points directly toward the z-axis.

This is a combination (i.e. a "superposition") of the first two examples:

- uniform circular motion in the x- & y-coordinates
- constant linear motion in the z-coordinate.

Newton's Second Law

If a force $\vec{F}$ acts on a particle of mass m then it causes an acceleration $\vec{a}$ such that

$$\boxed{\vec{F} = m\vec{a}}$$

Here $\vec{F}$ & $\vec{a}$ are vectors & m is a (positive) scalar.

The typical physics problem:

At each point $\vec{p} \in \mathbb{R}^3$ we have some force $\vec{F}(\vec{p})$. [We can think of $\vec{F}$ as a function $\mathbb{R}^3 \to \mathbb{R}^3$, called a "force field".]

If we place a particle of mass m in the force field at point $\vec{p}$, our goal is to predict the trajectory of the particle. That is, we want to find the unique parametrized path

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

such that

- $\vec{r}(0) = \vec{p}$

[The position at time 0 is $\vec{p}$]

- $\vec{F}(\vec{r}(t)) = m\,\vec{r}\,''(t)$

[The force felt by the particle at time t satisfies $\vec{F} = m\vec{a}$.]

Next time we'll compute some examples where $\vec{F}$ is the force of gravity near the surface of the Earth, which is roughly constant:

$$\vec{F} \approx \langle 0, 0, -9.81 \underbrace{\text{kg}\cdot\text{m}/\text{sec}^2}_{\text{units of force are called ``Newtons''}} \rangle$$